Logical Truth, Contradictions, Inconsistency, and Logical Equivalence

Logical Truth

- The semantical concepts of logical truth, contradiction, inconsistency, and logical equivalence in Predicate Logic are straightforward adaptations of the corresponding concepts in Sentence Logic.
- A closed sentence $X$ of Predicate Logic is **logically true**, $\models X$, if and only if $X$ is true in all interpretations.
- The logical truth of a sentence is proved directly using general reasoning in semantics.
- Given soundness, one can also prove the logical truth of a sentence $X$ by providing a derivation with no premises.
- The result of the derivation is that $X$ is **theorem**, $\vdash X$.

An Example

- $\models (\forall x)Fx \supset (\exists x)Fx$.
  - Suppose $d$ satisfies ‘$(\forall x)Fx$’.
  - Then all $x$-variants of $d$ satisfy ‘$Fx$’.
  - Since the domain $D$ is non-empty, some $x$-variant of $d$ satisfies ‘$Fx$’.
  - So $d$ satisfies ‘$(\exists x)Fx$’
  - Therefore $d$ satisfies ‘$(\forall x)Fx \supset (\exists x)Fx$’, QED.
- $\vdash (\forall x)Fx \supset (\exists x)Fx$.

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<td>$(\forall x)Fx$</td>
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<td>$Fa$</td>
<td>$1 \forall E$</td>
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Contradictions

- A closed sentence $X$ of Predicate Logic is a **contradiction** if and only if $X$ is false in all interpretations.
- A sentence $X$ is false in all interpretations if and only if its negation $\sim X$ is true on all interpretations.
- Therefore, one may directly demonstrate that a sentence is a contradiction by proving that its negation is a logical truth.
- If the $\sim X$ of a sentence is a logical truth, then given completeness, it is a theorem, and hence $\sim X$ can be derived from no premises.
• If a sentence \( X \) is such that if it is true in any interpretation, both \( Y \) and \( \sim Y \) are true in that interpretation, then \( X \) cannot be true on any interpretation.

• Given soundness, it follows that if \( Y \) and \( \sim Y \) are derivable from \( X \), then \( X \) is a contradiction.

An Example

• ‘(\( \forall x \))(Fx \& \sim Fx)’ is a contradiction.
  – Suppose that a variable assignment \( d \) satisfies ‘(\( \forall x \))(Fx \& \sim Fx)’.
  – Then all \( x \)-variants \( d[u/x] \) of \( d \) satisfy ‘Fx \& \sim Fx’.
  – Then \( d[u/x] \) satisfies ‘Fx’.
  – Then \( d[u/x] \) satisfies ‘\sim Fx’.
  – Then \( d[u/x] \) does not satisfy ‘Fx’, a contradiction.
  – Therefore, no variable assignment \( d \) satisfies ‘(\( \forall x \))(Fx \& \sim Fx)’, QED.

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Inconsistent Sets of Sentences

• A set of closed sentences of Predicate Logic is **consistent** if and only if there is an interpretation (a **model**) which makes all the sentences in the set true.

• A set of closed sentences of Predicate Logic is **inconsistent** just in case it is not consistent.

• Therefore, a set of closed sentences of Predicate Logic is inconsistent just in case it has no models.

• It follows from these definitions and that of a contradiction that a finite collection of sentences is inconsistent if and only if the conjunction of the sentences is a contradiction.
  – There is no model for a set of sentences \( X \) if and only if in every interpretation, each of the sentences of \( X \) is false.
  – This holds if and only if in every interpretation, the conjunction of the sentences of \( X \) is false.
  – This holds if and only if the conjunction of the sentences of \( X \) is a contradiction, QED.
Demonstrating Inconsistency

- The consistency of a set of closed sentences can be demonstrated by providing a single interpretation which makes all the sentences in the set true.
- A direct demonstration of inconsistency requires general reasoning.
- Inconsistency can be proved indirectly by either of two ways.
  - Derive a contradiction from the set of inconsistent sentences taken as premises.
  - Derive the negation of the conjunction of the sentences from no premises.

An Example

- \{ '(\forall x)Fx', '~(\exists x)Fx' \} is inconsistent.
  - Suppose there is an interpretation which makes both '(\forall x)Fx' and '~(\exists x)Fx' true.
  - Then for a given variable assignment d, d satisfies both '(\forall x)Fx' and '~(\exists x)Fx'.
  - Therefore, all x-variants d[u/x] of d satisfy 'Fx'.
  - So d satisfies '(\exists x)Fx'.
  - It also follows that d satisfies '~(\exists x)Fx', which yields a contradiction.
  - So, there is no interpretation which makes both '(\forall x)Fx', '~(\exists x)Fx' true, QED.

The Example Continued

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Logical Equivalence

- Two closed sentences of Predicate Logic are **logically equivalent** if and only if they have the same truth value in all interpretations.
- The logical equivalence of X and Y holds as well when X is true in all interpretations where Y is true, and Y is true in all interpretations where X is true.
- Alternatively, two sentences X and Y are logically equivalent just in case their bicondition X \iff Y is a logical truth.
• Logical equivalence is demonstrated directly through general reasoning.
• It is proved indirectly with two derivations, each having one of the sentences as a premise and the other as a conclusion.