Meta-Logic of Predicate Logic

Derivability and Validity

- An argument of Predicate Logic $Z$, therefore $X$ (or $Z \because X$) is valid if and only if in every interpretation in which all the members of $Z$ are true, $X$ is also true.
  
  $\{ (\forall x)(Fx \supset Gx), Fa \} \vdash Ga$

- $X$ is derivable from premises $Z$ if and only if there is a proof of $X$ from only premises $Z$ using the rules of inference.

1. $(\forall x)(Fx \supset Gx)$  
2. $Fa$  
3. $Fa \supset Ga$  
4. $Ga$  

Soundness and Completeness

- $Z \vdash X$ if and only if $X$ is derivable from $Z$.
  
  - ‘$\vdash$’ is a metalogical symbol known as the ‘turnstyle’.
  
  - It indicates a purely syntactical relation between the members of $Z$ and $X$.

- $Z \models X$ if and only if the argument $Z \setminus X$ is valid.
  
  - ‘$\models$’ is a metalogical symbol known as the ‘double turnstyle’.
  
  - It indicates a semantical relation between the members of $Z$ and $X$.

- A system of formal proof is sound if and only if for all $Z$ and all $X$, if $Z \vdash X$, then $Z \models X$.

- A system of formal proof is complete if and only if for all $Z$ and all $X$, if $Z \models X$, then $Z \vdash X$ (proved by Kurt Gödel in 1930).

Undecidability

- Given that a system of formal proof is complete, there is a proof for every valid argument.

- In Predicate Logic, there is no purely syntactical way to determine whether a proof for any given argument exists or semantical way to determine whether any given argument is valid.

- There is no mechanical decision procedure for determining derivability or validity in a finite number of steps.
• Thus Predicate Logic is **undecidable** (proved by Alonzo Church in 1936).

• However, various fragments of Predicate Logic are decidable.
  – For example, the set of sentences of Predicate Logic in which no predicate has more than one argument.

**Second-Order Predicate Logic**

• Predicate Logic, originally proposed by Frege, can be extended by allowing quantification over predicates.
  – For example, we might formulate an axiom of identity.
  – \((\forall x)(\forall y)(x = y \equiv (\forall P)(Px \equiv Py))\).

• Semantically, the quantifiers would range over properties or sets, rather than the objects in the domain over which first-order quantifiers range.

• Second-order logic cannot be given a proof system which is sound and complete relative to its interpretation, though fragments of it can be given a sound and complete proof system.

• As with first-order logic, the fragment of second-order logic consisting of none but one-place predicates is decidable.

• It is possible to construct logics of higher, even infinite, order.