1 Syntax of MSL

Modal Sentential Logic (MSL) is a formal language which is an extension of Sentential Logic (SL). All expressions of SL are expressions of MSL and all sentences of SL are sentences of MSL. The semantics for MSL is generated by adding to the semantics for SL new semantical machinery to accommodate expressions proper to MSL. Derivations using MSL sentences contain rules of inference of the non-modal sentential derivational system SL and additional new inference rules to handle the additional expressions of MSL.

MSL extends SL syntactically by adding as expressions modal operators. Most versions of modal logic syntax add at least one of two one-place operators: the box ‘□’ and the diamond ‘♦’. In the original work on MSL by C. I. Lewis, two two-place operators, the fish-hook ‘≺’ and the circle ‘◦’, were given prominence.  

A sentence whose main logical operator is a modal operator will be called a modal sentence. Informal interpretations for the four kinds of modal sentences formed with the four kinds of modal operators just mentioned will be given in the next section.

Four rules of sentence-formation are added to the syntax of SL to obtain the syntax of MSL. The meta-linguistic variables ‘α’, ‘β’, and ‘γ’, with or without subscripts, are intended to refer to sentences of the language under consideration (here, MSL).  

1Lewis’s early work is presented in Chapter V, “The System of Strict Implication”, of his A Survey of Symbolic Logic (Berkeley, 1918). The chapter was excised from a subsequent edition because it contained a deep substantive error.

2Expressions formed from these meta-variables and expressions of MSL will not be placed in single quotation marks unless we are referring to them as elements of the meta-language. Operators, including modal operators, will be used as names of themselves.
Syntax of MSL

If $\alpha$ is a sentence letter of SL, then $\alpha$ is a sentence of MSL.

If $\alpha$ is a sentence of MSL, then $\sim \alpha$ is a sentence of MSL.

If $\alpha$ and $\beta$ are sentences of MSL, then $\alpha \lor \beta$ is a sentence of MSL.

If $\alpha$ and $\beta$ are sentences of MSL, then $\alpha \supset \beta$ is a sentence of MSL.

If $\alpha$ and $\beta$ are sentences of MSL, then $\alpha \equiv \beta$ is a sentence of MSL.

If $\alpha$ is a sentence of MSL, then $\Box \alpha$ is a sentence of MSL.

If $\alpha$ is a sentence of MSL, then $\Diamond \alpha$ is a sentence of MSL.

If $\alpha$ and $\beta$ are sentences of MSL, then $\alpha \preceq \beta$ is a sentence of MSL.

If $\alpha$ and $\beta$ are sentences of MSL, then $\alpha \circ \beta$ is a sentence of MSL.

Nothing else is a sentence of MSL.

Any modal operator could be made primitive by defining the three others in terms of it. Here are some combinations that are frequently used.\(^3\)

**Definitions of Modal Operators**

- $\Box \alpha =_{DF} \sim \Diamond \sim \alpha$
- $\Diamond \alpha =_{DF} \sim \Box \sim \alpha$
- $\alpha \preceq \beta =_{DF} \Diamond (\alpha \& \sim \beta)$
- $\alpha \circ \beta =_{DF} \Diamond (\alpha \& \beta)$

**Exercise.** Show how to define the fish-hook in terms of the circle. For a much more difficult challenge, try to define either one-place operator in terms of the fish-hook.

## 2 Informal Semantics for MSL

In themselves, the modal operators and sentences formed from them have no meaning. They are now given precise meaning by formal semantical rules. In the period before contemporary formal semantics was developed, modal sentences were given informal interpretations in English. In the most common informal interpretation, the box is supposed to represent necessary truth.\(^4\) For example, we might wish to assert the necessary truth of the truth-functionally true SL sentence '$\sim (P \& \sim P)$' by placing it in the scope of a box:

$$\Box \sim (P \& \sim P).$$

The diamond generally is taken to represent possibility. A simple example is a case where it is it is affixed to the left of a sentence letter such a ‘Q’. All

---

\(^3\)The meta-linguistic expression '$=_{DF}$' means 'is defined as'.

\(^4\)It might also represent knowledge, belief, obligation, future times, and other notions. These informal interpretations will be discussed extensively below.

\(^5\)A sentence is truth-functionally true just in case it is true on all rows of its truth-table. It is truth-functionally false just in case it is false on all rows of its truth-table. And it is truth-functionally indeterminate if it is true on at least one row of its truth-table and false on at least one.
sentence letters are not truth-functionally false, so we may wish to say that ‘Q’
is at least possibly true, which can be expressed as follows:

\( \Diamond Q \).

Note that the syntax of MSL allows that a modal operator may occur in a
position other than that of the main logical operator of a sentence. On a row
of a truth-table on which ‘Q’ is false, its negation, ‘\(~Q\)’, is true. So we might
say that the negation is possibly true, in which case ‘Q’ is possibly false. We
can state that ‘Q’ is possibly true and possibly false in this way:

\( \Diamond Q \land \Diamond \sim Q \).

In such a case, the sentence is said to be\emph{ contingent}.

The two-place fish-hook operator is intended to represent “strict implic-
ation”. This is a relation between two sentences when it is impossible for the
first to be true and the second false. When the truth-values are assigned as in the
semantics for SL, this amounts to the same thing as saying that the first
truth-functionally entails the second.\footnote{A set of sentences \( \Gamma \) truth-functionally entails a sentence \( \alpha \) just in case there is no assign-
ment of truth-values to the sentence letters such that all the members of \( \Gamma \) are true and \( \alpha \) is false. In the semantics for SL, this definition is equivalent
to saying that in every assignment of truth-values to the sentences letters such that all the members of \( \Gamma \) are true, \( \alpha \) is true. In
some semantical systems for Modal Predicate Logic, this equivalence breaks down, as will be
seen. We shall represent individual sentences of \( \Gamma \) as \( \gamma_1, \ldots, \gamma_n \).}
In SL, the sentence ‘P & Q’ truth-functionally
entails ‘P’. So we may wish to write:

\( (P \land Q) \prec P \).

The circle operator is intended to indicate that one sentence is consistent
with another.\footnote{In the semantics for SL, two sentences are consistent when there is a row on a truth-table
which makes them both true.} For example, the sentence ‘P’ can be given the value true and
‘Q’ given the value true on the same row of a truth-table. So we might wish to
write:

\( P \circ Q \).

In the next section, these informal notions will be given a rigorous treatment.
But it can be seen already that the modal operators can be used to express in
MSL some of the most important semantical properties and relations of SL
sentences.\footnote{As will be seen, they can also express semantical properties and relations of MSL sentences.} This was Lewis’s original goal in developing modal logic.

3 Basic Formal Semantics for MSL

The semantics for modal sentential logic is a generalization of the semantics for
non-modal sentential logic. Sentential logic sentences are assigned truth-values
by an\emph{ interpretation}. An SL-interpretation \( \mathbf{I} \) is nothing more than a collective
\emph{truth-value assignment} (TVA) to all the sentence-letters of SL. The TVA made
by an interpretation gives to each sentence-letter either the value \( T \) or the value
F. (We shall call such an assignment a complete TVA. In general, we will be working with partial truth-value assignments, which leave out the values of expressions of SL or MSL not found in the sentences under evaluation.)

There are well-known rules for the assignment of truth-values to a non-atomic sentence based on the truth-values of its components. These rules will be given after we develop the semantics for the modal operators. In the meantime, we can say that from the assignment made by the interpretation, the value of the sentence results directly, according the rules for the assignment of truth-values for the sentential operators. Thus if an interpretation assigns to the sentence-letter ‘P’ the value T, then the value of ‘¬P’ is F, etc. We will say that in such a case, the operator and the semantical rule governing it are directly truth-functional.

For modal sentences, the situation is different: the standard semantical rules for modal operators are not directly truth-functional. We cannot obtain the value of □α on an interpretation simply from a single value for α given by the interpretation. The one-place modal operator does not function semantically in the way the one-place negation operator does. Nonetheless, the determination of the value of the modal sentence □α is ultimately based on truth-values assigned by the given interpretation to α, and so semantics for modal sentential logic is indirectly truth-functional in a way that will be described at the end of this section.

We may depict in the standard truth-table format a partial TVA for an SL sentence. So, for example, if an interpretation I assigns T to ‘P’ and F to ‘Q’, we have a table that looks like this.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

And by the semantical rules for conjunction we have the familiar:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P &amp; Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

where the value of the conjunction is F because the assignment to one of its conjuncts is F.

Note that this determination is entirely self-contained. No reference is made to any other TVA but the one which assigns T to ‘P’ and F to ‘Q’. To make another assignment would be to give a different interpretation, since there is no distinction in the semantics for SL between an interpretation and the assignment it makes. The truth and falsehood of a sentence is relative to a single given truth-value assignment, and hence to a single given interpretation.

A different interpretation with a different TVA gives a different result for the sentence ‘P & Q.’

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P &amp; Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
Once again, no reference is made to any other truth-value assignment. It is as if each TVA represents a “world” of its own.

It will be convenient in developing the semantics for MSL to think of the truth-value assignment given by an interpretation I as made by a valuation-function v_I. To state that I assigns the value T to α we write:

\[ v_I(\alpha) = T. \]

We can express the end-result of the preceding truth-table in functional notation, as:

\[ v_I(P \& Q) = T. \]

Although the determination of the truth-values of sentences of SL depends entirely on I and its TVA, other semantic properties of the sentence can be determined only by looking at more than one interpretation of it. Whether a sentence is truth-functionally true, false or indeterminate, for example, requires that we look at the values for the sentence under different TVAs. Thus the sentence ‘P & Q’ is truth-functionally indeterminate, as we have already shown by producing a TVA under which it is true and a different one under which it is false.

Now let us consider the disjunction ‘P ∨ ∼P,’ which is truth-functionally true. We can say this because there are two possible partial TVAs making an assignment to ‘P,’ producing the following two tables.\(^9\)

<table>
<thead>
<tr>
<th></th>
<th>(P)</th>
<th>(P ∨ ∼P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Granting that the assignments made to any other sentence letters are irrelevant to the truth or falsehood of ‘P ∨ ∼P,’ we can say that it is true on all interpretations (which just are truth-value assignments), and so it is truth-functionally true.

Truth-functional truth and the related notions are concepts which are not expressible in the syntax of SL itself, but only in the meta-language we use to talk about SL. The formal meta-linguistic notation for truth-functional truth, i.e.,

\( \models_{SL} P ∨ ∼P \)

uses a subscripted double-turnstile symbol that is not defined by the syntax of SL. But we can express something analogous in MSL:

\( □(P ∨ ∼P) \).

Modal sentential logic can be viewed as simulating, in the modal object-language

\(^9\)Of course, this result is usually depicted in a single table, but two are used here to indicate the fact that it is not a row on a truth-table by itself, but the matching of a truth-value to a sentence which gives the desired result.
MSL, meta-logical properties and relations. This is done by extending the object-language SL to include modal expressions whose truth-values on an interpretation mostly depend on multiple truth-value assignments. Such an extension allows the representation of semantical properties and relations that are more general than simple truth and falsehood.

Though there are several modal operators defined in the syntax of MSL, we shall limit our initial discussion to the one-place modal operators ‘□’ and ‘♦’. From a given sentence α we can form the necessity-sentence □α in MSL. We will express in the formal semantics the informal meaning of the necessity sentence according to which □α is true just in case α is necessarily true. A possibility-sentence ♦α will be true in the formal semantics just in case it is possible that α is true. (Again, it must be stressed that these modal operators can be given other readings.)

From a formal standpoint, to interpret necessity- and possibility-sentences we need a way of making reference to multiple truth-value assignments within a single interpretation. The semantics for Modal Sentential Logic does not identify interpretations with TVAs in the manner of the semantics for non-modal Sentential Logic. In the semantics for MSL, a single interpretation must be able to allow more than one distinct truth-value assignment to the sentence letters. This means that the application of an SL valuation-function to a sentence α, v_I(α), is not adequate for the semantics for MSL.

To accommodate multiple truth-value assignments, we require what are most commonly known as “possible worlds.” We may think of possible worlds as “locations” with respect to which truth-value assignments are made. Worlds will be indicated by the meta-variable w with or without an integer or lower-case italic alphabetic subscript. Thus, to build on our previous example, we could say that an interpretation I of ‘P’ contains two partial truth-value assignments. We can call the first partial TVA, which assigned ‘P’ the value T, an assignment at w_1, and the second partial TVA, which assigned ‘P’ the value F, an assignment at w_2.

\[
\begin{array}{c|c|c}
  w_1 & P & P \lor \neg P \\
  \hline
  T & T \\
\end{array}
\]

\[
\begin{array}{c|c|c}
  w_2 & P & P \lor \neg P \\
  \hline
  F & T \\
\end{array}
\]

We shall have recourse to alphabetic subscripts, such as w_i, when we wish to talk about arbitrary worlds, as will be done below.

Now we can extend the notion of a valuation function to make it adequate for the formal semantics of MSL. The formal semantics for MSL requires a two-place function v_I, which maps a pair consisting of a sentence and a world onto

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10They have also been called assignments, situations, states of affairs, indices, points, etc. Whatever their name, they are nothing more than reference points for assignments of truth-values to sentences.
truth-values, i.e., the set \{T, F\}. In the example just given, we can say with respect to our interpretation I:

\[ v_I(P, w_1) = T, \text{ and } \]
\[ v_I(P, w_2) = F. \]

However, for any value of w,
\[ v_I(P \lor \sim P, w) = T. \]

It might be helpful to recognize that we could have used two-place valuation functions in the semantics for sentential logic. That is, instead of subscripting the ‘v’ with a reference to the interpretation under which it makes its assignments, we could have made that reference the second argument of the function:
\[ v_I(\alpha) = v(\alpha, I). \]

But this notation would not have allowed the generalization to the semantics for modal logic, since it presupposes a one-to-one correspondence between interpretations and truth-value assignments.

At this point, we are close to being able to state the specific rule for the determination of the truth-values of necessity- and possibility-sentences at a world. But we require one more piece of machinery. We are to think of the possible worlds in an interpretation as ordered under a relation of accessibility.

We will say, for example, that on an interpretation I, a specific world, \( w_2 \), is accessible to a specific world, \( w_1 \). This will be stated formally as \( Rw_1w_2 \). Every interpretation requires both a non-empty set of worlds and a relation of accessibility among the members of the set of worlds. Together, the set of worlds and the accessibility relation constitute a frame. If we use ‘W’ to indicate a set of worlds, a frame \( Fr \) can be represented as an ordered pair, so that \( Fr = (W, R) \).\(^\text{11}\) We add a valuation function ‘v’ to a frame to get an interpretation: \( I = (W, R, v) \). The interpretation formed in this way is said to be based on the frame to which the valuation function is added. We will say that a world is in an interpretation I when it is a member of the set W, which is a member of I.

Now we can state the semantical rule or truth-definition for a sentence governed by the necessity operator. (Note that if a sentence is not assigned T, it is assigned F.)

\textbf{Truth-Definition for Necessity}
\[ v_I(\Box \alpha, w) = T \text{ iff } v_I(\alpha, w_i) = T \text{ at all worlds } w_i \text{ in } I \text{ such that } Rww_i. \]

The truth-condition for the possibility-sentence is similar:

\textbf{Truth-Definition for Possibility}
\[ v_I(\Diamond \alpha, w) = T \text{ iff } v_I(\alpha, w_i) = T \text{ at some world } w_i \text{ in } I \text{ such that } Rww_i. \]

And we are now in a position to give truth-definitions for the operators of SL.\(^\text{11}\)Ordered n-tuples are expressed with angled brackets and n elements, separated by commas.
Truth-Definitions for Non-Modal Operators

\[ v_I(\neg \alpha, w) = T \text{ if and only if } v_I(\alpha, w) = F. \]

\[ v_I(\alpha & \beta, w) = T \text{ if and only if } v_I(\alpha, w) = T \text{ and } v_I(\beta, w) = T. \]

\[ v_I(\alpha \lor \beta, w) = T \text{ if and only if } v_I(\alpha, w) = T \text{ or } v_I(\beta, w) = T. \]

\[ v_I(\alpha \implies \beta, w) = T \text{ if and only if } v_I(\alpha, w) = F \text{ or } v_I(\beta, w) = T. \]

\[ v_I(\alpha \equiv \beta, w) = T \text{ if and only if } v_I(\alpha, w) = v_I(\beta, w). \]

Given the truth-definitions for sentences of MSL, we can define meta-logical properties and relations of MSL sentences. Because different restrictions on the accessibility relation in frames will generate different systems of modal logic, we shall give formal definitions which are fully general.

One of the fundamental relations in our semantics is that of semantic entailment, which holds between a set of MSL sentences and a single sentence.\(^{12}\) In the semantics for non-modal Sentential Logic, a set of sentences \(\{\gamma_1, \cdots, \gamma_n\}\) semantically entails a sentence \(\alpha\) if and only if on all interpretations on which each sentence in the set is true, \(\alpha\) is true. Modal semantics requires that we expand this definition to include possible worlds, since a sentence only has a truth-value at a world. Thus we will say that the relation of semantic entailment holds when all the interpretations based on a given frame which make all the sentences \(\gamma_i\) of the set true at a world also make the sentence \(\alpha\) true at that world. The strict definition is as follows.

**Semantical Entailment in a Frame \(Fr\)**

\(\{\gamma_1, \cdots, \gamma_n\} \models_{Fr} \alpha\) just in case for any \(I\) based on \(Fr\) and any \(w\) in \(W\) in \(Fr\), if \(v_I(\gamma_1, w) = T\) and, \(\cdots\), and \(v_I(\gamma_n, w) = T\), then \(v_I(\alpha, w) = T\).

For example, for any frame \(Fr\), \(\{\neg P, P \lor Q\} \models_{Fr} Q\). On an arbitrary interpretation \(I\) and world \(w\) in \(I\), if both ‘\(P\)’ and ‘\(\neg P \lor Q\)’ are assigned \(T\) at \(w\), then ‘\(Q\)’ must be assigned \(T\) as well. This can be seen easily enough from a truth-table containing all the partial assignments to ‘\(P\)’ and ‘\(Q\)’. Generally, truth-functional entailment for SL carries over to entailment in all frames of MSL.

**Exercise.** Give an argument for why this is so.

The limiting case of entailment is validity. In Sentential Logic, a sentence is said to be truth-functionally valid just in case it is true on all interpretations. We can think of validity as semantic entailment by the empty set of sentences: \(\emptyset \models_{Fr} \alpha\). So the definition of validity in a frame will incorporate the consequent of the definition of entailment in a frame. A sentence \(\alpha\) is valid in a frame if and only if it is true at all worlds on all interpretations based on that frame.

**Validity in Frame \(Fr\)**

\(\models_{Fr} \alpha\) if for all \(w\) in \(W\) in \(Fr\), all \(I\) based on \(Fr\), and all \(v_I\) in \(I\), \(v_I(\alpha, w) = T\)

\(^{12}\) We shall for convenience refer to this relation simply as “entailment.” This technical notion must be distinguished from more vague, intuitive notions of entailment, which may not correspond to semantical entailment.
For example, \( \neg \neg P \) is valid any frame whatsoever. This is because the sentence is true at every world that could be in a frame, whether \( P \) is assigned True or False. What this really means is that the various sets of worlds to be found in different frames and the various interpretations based on those frames have no effect on the valuation of the sentence. So it is valid on any frame. We can generalize this reasoning to conclude that every truth-functionally true sentence of \( SL \) is valid in every frame for \( MSL \).

**Exercise:** Give a definition of *consistency in a frame* which corresponds to the notion of truth-functionally consistency. A set of sentences is truth-functionally consistent if and only if there is a row on a truth-table which makes all the sentences in that set true.

We shall put this machinery to work shortly. In the meantime, it may be useful to reflect on the nature of the accessibility relation. One way to think of it is as representing those situations that matter with respect to the modality in question. For example, we might think of something as necessary when it is inevitable in the sense that there is no alternative that matters to its being the case. The alternatives that matter might be thought of as being “accessible worlds”. Truth of the necessity-sentence \( \Box \alpha \) would require truth of the embedded sentence \( \alpha \) in all those worlds that matter. A possibility-sentence would indicate that its embedded sentence is true in at least one of the worlds that matters. We shall in what follows have much more to say about how accessibility captures our intuitions about the various kinds of modalities.

We are now finally in a position to see why the semantics for modal sentential logic is indirectly truth-functional. Although the truth of a necessity sentence (or any other modal sentence) is not directly a function of the assignments made to its components, it is a function of the assignments made to its components at the accessible possible worlds. And those assignments are ultimately based on truth-value assignments to the sentence-letters from which the sentence is composed.

### 4 The Basic Derivation Rules for *MSL*

The non-modal derivation system for Sentential Logic, \( SD \), is a reflection of the semantics for \( SL \). Essentially, writing an \( SL \) sentence on a line of a derivation is taking it to be true. So we can infer \( \alpha \) from \( \alpha \& \beta \) because if we take \( \alpha \& \beta \) to be true on a truth-value assignment, we must take \( \alpha \) to be true as well, by the semantical rules. When an assumption is made, the truth of the sentence is held provisionally. Discharging the assumption shows what is true given that the assumption is true.\(^{13}\)

The derivability relation, \( \{ \gamma_1, \ldots, \gamma_n \} \vdash_{SD} \alpha \), holds between a set of sentences and a sentence when each member of the set is an assumption and \( \alpha \) is a step, not in the scope of any undischarged assumption, conforming to \( SD \)

\(^{13}\) Sometimes other sentences are taken to be true as well, as with the disjunction in \( \vee \) Elimination.
rules. \(SD\) is sound with respect to the semantics for \(SL\). That is, if a sentence is derivable from a set of sentences, it is semantically entailed by that set of sentences.\(^{14}\)

**Soundness**

If \(\{\gamma_1, \cdots, \gamma_n\} \vdash_{SD} \alpha\), then \(\{\gamma_1, \cdots, \gamma_n\} \models_{SL} \alpha\).

Moreover, the converse, completeness, also holds. If a set of sentences entails a sentence, then there is a derivation of that sentence from all the members of the set as assumptions.

**Completeness**

If \(\{\gamma_1, \cdots, \gamma_n\} \models_{SL} \alpha\), then \(\{\gamma_1, \cdots, \gamma_n\} \vdash_{SD} \alpha\).

As we built the syntax and semantics for \(MSL\) from the syntax and semantics of \(SL\), we can build on \(SL\) derivation rules (the \(SD\) rules) to define derivations for \(MSL\). We want modal derivability to reflect exactly the relation of semantic entailment in the semantics for \(MSL\). The key is to generate rules of inference which reflect the semantical rules for the modal operators.

### 4.1 Derivation Rules for the ‘□’ Operator

We shall begin our treatment of the derivation rules with the ‘□’ operator, whose semantical rule is:

\[
v_I(\Box \alpha, w) = T \text{ iff } v_I(\alpha, w_i) = T \text{ at all worlds } w_i \text{ in } I \text{ such that } R_{ww_i}.
\]

As this rule is a meta-logical biconditional, it can be broken down into two elements. Let us first consider the conditional that gives a sufficient condition for the truth of a necessity-sentence at a world.

If \(v_I(\alpha, w_i) = T\) at all worlds \(w_i\) in \(I\) such that \(R_{ww_i}\), then \(v_I(\Box \alpha, w) = T\).

To establish that the consequent holds, one must show that \(\alpha\) holds at any arbitrary accessible world and then generalize to the conclusion that \(\alpha\) holds at all accessible worlds. To represent such a situation, we make use of the device of *restricted scope lines*, double vertical lines which graphically represent the relation of accessibility.

It is crucial that importation of earlier steps across restricted scope lines be limited. What is outside the restricted scope line reflects a truth-value assignment at a world that may be different from the one represented by the restricted scope line. Suppose a sentence \(\alpha\) has the value \(T\) at a world \(w\), and that \(R_{ww_1}\). We cannot in general expect that the value at \(w\) will be preserved at \(w_1\). If \(\alpha\) is a sentence letter such as ‘\(P\)’, it may well be true at \(w\) but false at \(w_1\). Since each step in a derivation is supposed to represent the assignment of truth to a sentence, we might go astray by writing down ‘\(P\)’ to the right of a restricted scope line when it occurs to the left. So we shall stipulate that no \(SD\) rule

\(^{14}\)We shall sometimes refer collectively to entailment and derivability as *consequence* relations.
(including Reiteration) may be applied across a restricted scope line, though any rule may be applied within a restricted scope line.

We can now state the derivation rule that reflects the kind of semantical reasoning used in establishing the truth of necessity-sentences at a world. It holds for all MSL derivation systems which conform to the semantics given in the last section. This is the rule of $\Box$ Introduction. A restricted scope line is ended, and a box is prefixed to the sentence on its last line.

$$
\begin{array}{c}
\Box \text{Introduction} \\
\hline \\
\hline
\alpha \\
\hline
\Box \alpha \Box
\end{array}
$$

Provided that no SD rule is applied across the restricted scope line.

To illustrate the use of this rule, we can take advantage of the fact that all the rules for SD apply to derivations carried out wholly within a restricted scope line. For example, we can derive $\Box \sim (P \& \sim P)$ using some rules of SD and $\Box$ Introduction.

1. $P \& \sim P$ Assumption
2. $P$ 1 &E
3. $\sim P$ 1 &E
4. $\sim (P \& \sim P)$ 1-3 $\sim$I
5. $\Box \sim (P \& \sim P)$ 1-4 $\Box$I

Derivations of this kind are the only ones that can be carried out using $\Box$ Introduction alone. What is needed to allow further derivations is a rule which will allow us to import information across restricted scope lines. Such a rule will reflect the fact that the scope line represents the relation of accessibility. The source of the rule is to be found in the other half of the semantical rule for the $\Box$.

If $v_I(\Box \alpha, w) = T$, then $v_I(\alpha, w_i) = T$ at all worlds $w_i$ in $I$ such that $Rww_i$.

Given that the restricted scope line is supposed to represent an arbitrary accessible world, the fact that $\Box \alpha$ is true at all accessible worlds implies that it is true at the arbitrary accessible world. So where we already have $\Box \alpha$ written

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15 These systems are called “normal” systems of Modal Sentential Logic. The way in which derivational systems conform to semantical systems will be explained in the context of system $K$, which we will examine in the next section.

16 We will not here give a $\Box$ Elimination rule. One such rule would allow the removal of the $\Box$ operator from $\Box \alpha$, so that $\alpha$ can be written down. But nothing about the semantical rule for the $\Box$ requires that if $v_I(\Box \alpha, w) = T$, then $v_I(\alpha, w) = T$. Such a rule will be forthcoming when we consider semantical systems strong enough to support it.
down, we can import \( \alpha \) across the restricted scope line.\(^{17}\) This rule is known as *Strict Reiteration*. (This rule might be thought of as a \( \Box \) Elimination rule, but we will reserve that denomination for another rule to be introduced in the next chapter.)

\[
\text{Strict Reiteration}
\]

\[
\begin{array}{c}
\Box \alpha \\
\vdots \\
\vdots \\
\alpha \text{ SR}
\end{array}
\]

Provided that only one restricted scope line is crossed

Strict Reiteration works in tandem with \( \Box \) Introduction. After Strict Reiteration is used, there remains an open restricted scope line which needs to be finished off with \( \Box \) Introduction to complete the derivation or sub-derivation. And \( \Box \)I is what ends the restricted scope line. An example will illustrate this. We may derive ‘\( \Box Q \)’ from ‘\( \Box P \)’ and ‘\( \Box (P \Rightarrow Q) \)’.

\begin{tabular}{c|l}
1 & \Box P & Assumption \\
2 & \Box (P \Rightarrow Q) & Assumption \\
3 & P & 1 SR \\
4 & P \Rightarrow Q & 2 SR \\
5 & Q & 3\, 4 \Rightarrow E \\
6 & \Box Q & 3\, 5 \Box I
\end{tabular}

This derivation parallels semantical reasoning. Suppose that \( v(\Box P, w) = T \) and \( v(\Box (P \Rightarrow Q), w) = T \). (Lines 1 and 2.) Then at any accessible world \( w_i \), \( v(P, w_i) = T \) and \( v(P \Rightarrow Q, w_i) = T \). (Lines 3 and 4.) By the rule for the ‘\( \Rightarrow \)’, \( v(Q, w_i) = T \). (Line 5.) Since the choice of \( w_i \) is arbitrary, at any accessible world, ‘\( Q \)’ is assigned the value \( T \). Therefore, \( v(\Box Q, w) = T \).\(^{18}\)

The derivation also closely resembles a Predicate Logic derivation that captures the interplay of quantifiers in the semantical argument.

\begin{tabular}{c|l}
1 & (\forall x)F x & Assumption \\
2 & (\forall x)(F x \Rightarrow G x) & Assumption \\
3 & F a & 1 \forall E \\
4 & F a \Rightarrow G a & 2 \forall E \\
5 & G a & 3 \, 4 \Rightarrow E \\
6 & (\forall x)G x & 3\, 5 \forall I
\end{tabular}

The \( \Box \) Introduction rule parallels \( \forall \) Introduction, and Strict Reiteration resembles \( \forall \) Elimination.

\(^{17}\)Note that only one restricted scope line may be crossed, since the semantical rule allows only truth at all worlds accessible to the given world. Also it is assumed that \( \Box \alpha \) occurs outside the scope of all discharged assumptions.

\(^{18}\)Aristotle is said to have counted something like this inference as valid. See *Prior Analytics*, Book I, Chapter 15.
4.2 Derivation Rules for ‘◊’ Operator

A sure way to a complete set of rules for the ‘◊’ operator is to take negation into account. Consider one of the definitions of possibility in terms of necessity given in Section 1.

$$\square \alpha =_{Df} \sim \diamond \sim \alpha$$

With this definition in hand, we can re-write the rules of Strict Reiteration and □ Introduction. First, Strict Reiteration would be definitionally equivalent to this pattern (with a negated sentence being strictly reiterated).

$$\sim \diamond \sim \sim \alpha$$

Given the equivalence of \(\sim \sim \alpha\) to \(\alpha\), we get a variant of Strict Reiteration for negated possibility: SR(\(\sim \diamond\)).

**Strict Reiteration for \(\sim \diamond\)**

$$\begin{array}{c}
\sim \diamond \alpha \\
\vdots \\
\sim \alpha \\
\end{array}$$

Provided that only one restricted scope line is crossed

The soundness of the rule can be seen from this semantical argument. If \(v_I(\sim \diamond \alpha, w) = T\), then \(v_I(\diamond \alpha, w) = F\), so there is no accessible world \(w\), at which \(v_I(\alpha, w_I) = T\). So given an accessible world, we must assign \(\alpha\) the value \(F\) there, in which case the value of \(\sim \alpha\) must be \(T\), which was to be shown.

The same sort of technique that produced the Strict Reiteration rule can be used to get a negated-possibility rule corresponding to □ Introduction. By substitution according to the definition, we have:

$$\begin{array}{c}
\sim \diamond \alpha \\
\vdots \\
\sim \alpha \\
\sim \diamond \sim \sim \alpha \\
\end{array}$$

and again, given a Double Negation substitution we get an Introduction rule.

$$\sim \diamond \textbf{Introduction}$$
Provided that no SD rule is applied across the restricted scope line

This rule is also sound, given the basic semantics. Suppose an arbitrary world \( w_i \) is accessible to a world \( w \). If a sentence \( \sim \alpha \) is assigned \( T \) at \( w_i \), then \( \alpha \) is assigned \( F \) at that world. Since \( w_i \) is arbitrary, \( \alpha \) is assigned \( F \) at all worlds accessible to \( w \). Thus \( \sim \Box \alpha \) is assigned \( T \) at \( w \), which was to be shown.

With these two rules we can derive the definitional equivalents of any necessity-sentence that can be derived using the necessity rules. For example, we have seen that from \( \Box P \) and \( \Box (P \supset Q) \) we can derive \( \Box Q \). This is equivalent to deriving \( \sim \Diamond P \) from \( \sim \Diamond Q \) and \( \sim \Diamond (P \& \sim Q) \). (The second premise can also be read as \( P \prec Q \).)

**Exercise.** Explain why these two derivations are equivalent.

---

1 \( \sim \Diamond Q \) Assumption
2 \( \sim \Diamond (P \& \sim Q) \) Assumption
3 \( \sim Q \) 1 SR(\( \sim \Diamond \))
4 \( P \) Assumption
5 \( P \& \sim Q \) 3 & I
6 \( \sim (P \& \sim Q) \) 2 SR(\( \sim \Diamond \))
7 \( \sim P \) 4-6 ~ I
8 \( \sim \Diamond P \) 3-7 ~ I

Another sample derivation shows the \( \sim \Diamond \) Introduction rule working by itself.

We can derive \( \sim \Diamond (P \& \sim P) \) from no undischarged assumptions, showing that it is a theorem of Modal Sentential Logic.

---

1 \( P \& \sim P \) Assumption
2 \( P \) 1 & E
3 \( \sim P \) 1 & E
4 \( \sim (P \& \sim P) \) 1-3 ~ I
5 \( \sim \Diamond (P \& \sim P) \) 1-4 ~ I

Despite their correspondence to the necessity rules, these two rules seem less than satisfactory because they involve a non-modal operator. They are not pure possibility rules. We shall give a derived pure possibility rule shortly. This rule was given by Fitch in the original adaptation of derivability rules to modal logic.\(^{19}\)

\(^{19}\)It is the opinion of the author that no combination of pure possibility rules will be complete relative to the basic modal semantics. In particular, the theorem just demonstrated seems not to be derivable without impossibility rules. It is interesting to note that in C.I. Lewis’s original formulations of axiomatic modal logic, the primitive modal operator was a symbol for impossibility. Other modal operators were defined in terms of impossibility. Lewis did not define a necessity operator, but it can be defined in terms of impossibility.
The Elimination rule for possibility works somewhat differently from the Strict Reiteration and the two Introduction rules we have provided. The clue for its structure is found in the fact that the semantical rule for the ‘◊’ involves an existential rather than a universal quantifier.

\[ v_I(\diamond \alpha, w) = T \iff v_I(\alpha, w_i) = T \text{ at some world } w_i \text{ in } I \text{ such that } Rww_i. \]

Consider one half of the rule:

If \[ v_I(\diamond \alpha, w) = T \], then \[ v_I(\alpha, w_i) = T \text{ at some world } w_i \text{ in } I \text{ such that } Rww_i. \]

When we represent an arbitrary accessible world by a restricted scope line, we cannot be sure that such a world is one at which \( \alpha \) is true. On an analogy with the elimination rule for the existential quantifier in Predicate Logic, the only thing we can do is to consider what would happen if the sentence were true at a possible world.

This is how we reason semantically. We say that because \( \diamond \alpha \) is true at a world, \( \alpha \) is true at some accessible world. Then we suppose that we are considering that world. If the truth of \( \beta \) is established at that world, then, given the assumption that the world is accessible, we conclude that \( \diamond \beta \) is true at the original world.

To represent this situation, we shall use a hypothetical restricted scope line, which will make a Modal Assumption. This line looks just like an SD assumption line, except that it is double.

\[ \diamond \text{ Elimination} \]

\[ \begin{array}{c}
\diamond \alpha \\
\alpha \\
\\
\beta \\
\diamond \beta \quad \diamond E
\end{array} \]

Provided that only one restricted scope line is crossed in making the assumption

A hypothetical restricted scope line functions like a regular restricted scope line in that it allows the use of SR across it.

But it should be noted that a hypothetical restricted scope line is a special bit of notation and always requires an assumption which is embedded in a possibility-sentence outside it scope.\(^{20}\) It is not the same as a restricted scope line with an assumption made inside it. A non-hypothetical restricted scope line allows the use of \( \square \) Introduction or \( \sim \diamond \) Introduction only.

One might be tempted to think that we could use a hypothetical scope line with an “empty” assumption, which would allow us to derive a theorem. But even if a theorem were derived inside the hypothetical restricted scope line, it

\(^{20}\) The possibility sentence must stand outside the scope of all discharged assumptions.
could not be brought out. This is because in the basic semantics some frames will contain worlds to which there is no world accessible. This means that we do not want to be able to derive even the possibility $\alpha$, where $\alpha$ is a theorem, unless we are working with the supposition that there are accessible worlds. The presence of the possibility sentence indicates that there is an accessible world at which it is true.21

This rule is a derived rule given the two Impossibility rules just stated. That is, if we assume we have derived $\Box \alpha$ and that we can derive $\beta$ from $\alpha$ within a restricted scope line with ‘$\alpha$’ as an assumption, we can derive $\Box \beta$.

\[ \begin{array}{c}
\Box \alpha \\
\sim \Box \beta \\
\alpha \\
\beta \\
\sim \beta \\
\sim \alpha \\
\sim \Box \alpha \\
\sim \Box \beta \\
\end{array} \quad SR(\sim \Box) \]

To illustrate the use of this rule, we will derive ‘$\Box Q$’ from ‘$\Box P$’ and ‘$\Box(P \supset Q)$’.22

\begin{array}{c|c}
1 & \Box P \text{ Assumption} \\
2 & \Box(P \supset Q) \text{ Assumption} \\
3 & \Box P \text{ Modal Assumption} \\
4 & \Box (P \supset Q) \text{ 2 SR} \\
5 & \Box Q \text{ 3 4 } \Box E \\
6 & \Box Q \text{ 1 3-5 } \Box E \\
\end{array}

This derivation reflects the argument for semantical entailment. If we assume that an interpretation $I$ assigns the two sentences ‘$\Box P$’ and ‘$\Box(P \supset Q)$’ the value $T$ at a world $w$, there is a world $w_1$ accessible to $w$ such that $v_I(P,w_1) = T$. Since ‘$\Box(P \supset Q)$’ is assigned $T$ at all accessible worlds, ‘$P \supset Q$’ is assigned $T$ at $w_1$. By the semantical rule for the ‘$\supset$’, ‘$Q$’ is assigned $T$ at $w_1$ as well. And since there is a world accessible to $w$ at which ‘$Q$’ is assigned $T$, ‘$\Box Q$’ is assigned $T$ at $w$ as well.

Note the similarity of this derivation to one that would establish that \{$(\exists x)Fx$, $(\forall x)(Fx \supset Gx)$\} $\vdash PD(\exists x)Gx$ in the derivation system $PD$ for Predicate Logic.

---

21 A $\Box$ Elimination rule will require stronger semantical rules than are provided by the basic semantics for $MSL$.

22 Aristotle is said to have counted something like this inference as valid. See Prior Analytics, Book I, Chapter 15.
This completes the exposition of basic semantical and derivational rules for our study of modal sentential logic. We shall now look at the system K which results from the use of just these rules. In the next chapter, we will turn to a number of other systems that result from strengthening the rules in various ways.

5 The System K

The semantical and derivational rules given in the last two sections are the basis for a number of systems of Modal Sentential Logic. We shall complete this chapter with a discussion of the weakest system K. In the next chapter, we shall examine systems stronger than K, i.e., systems which incorporate all the elements of K and support consequence-relations not found in K.

Recall that modal semantics is based on frames, each of which contains a set $W$ of possible worlds and a relation $R$ of accessibility defined on those worlds. An interpretation $I$ based on a frame adds to it a valuation function $v_I$ from sentences of MSL and worlds to truth-values: $I = \langle W, R, v \rangle$. The only restriction on $W$ is that there be at least one world.

Now consider all frames whatsoever, including frames in which the accessibility relation does not hold at all. We shall say in this case that the accessibility relation is unrestricted relative to the set of all frames, i.e., no world need be accessible to any other world. A K-frame is a member of the set of all frames. (In other words, every frame is a K-frame.) Then the accessibility relation is unrestricted relative to the set of all K-frames. Other semantical systems using the same semantical rules will be generated by placing restrictions on the accessibility relation.

The set of all interpretations based on K-frames (K-interpretations), along with the basic semantical rules for the sentences of MSL, constitutes the semantical system K. A sentence $\alpha$ will be said to be K-entailed by a set of sentences $\Gamma$, $\Gamma \models_K \alpha$, when it is entailed by $\Gamma$ on all K-frames (which is to say, all frames). A sentence is K-valid, $\models_K \alpha$, when it is true at all worlds on every interpretation based on a K frame.

The derivational system KD consists of the SD rules together with the rules of Strict Reiteration, $\Box$ Introduction, Strict Reiteration for Impossibility, Impossibility Introduction and $\Diamond$ Introduction. Other derivational systems are

\[\begin{array}{ll}
1 & (\exists x)Fx \quad \text{Assumption} \\
2 & (\forall x)(Fx \supset Gx) \quad \text{Assumption} \\
3 & Fa \quad \text{Assumption} \\
4 & Fa \supset Ga \quad 2 \forall E \\
5 & Ga \quad 3 \ 4 \supset E \\
6 & (\exists x)Gx \quad 5 \exists I \\
7 & (\exists x)Gx \quad 1 \ 3-6 \exists E
\end{array}\]
generated by adding to or strengthening the rules for $KD$. A sentence $\alpha$ is
$KD$-derivable from a set of sentences $\Gamma$, $\Gamma \vdash_{KD} \alpha$, just in case it is derivable
from these rules from the members of $\Gamma$ with no undischarged assumptions. A sentence
$\alpha$ is a $KD$-theorem, $\vdash_{KD} \alpha$, if it is derivable by these rules from no
undischarged assumptions. We assert without proof that the derivational rules
are sound and complete with respect to the semantics for $K$.

**Soundness and Completeness (for $K$)**

$\{\gamma_1, \cdots, \gamma_n\} \vdash_{KD} \alpha$ if and only if $\{\gamma_1, \cdots, \gamma_n\} \models_{K} \alpha$.

We shall frequently use the derivational system to show semantic entailment
and validity. It is easier to use than the semantical system, and derivations can
be converted automatically into semantical arguments. To show non-entailment,
non-validity and other properties that do not require the kind of general reason-
ing we have been using to establish entailment and validity, we shall construct
specific interpretations. These interpretations cannot be converted into deriva-
tions, as these properties cannot be established by derivation.\(^{24}\)

### 5.1 Closure

In a previous example, we proved that ‘$\Box Q$’ is both semantically entailed by
and derivable from the sentences ‘$\Box P$’ and ‘$\Box (P \supset Q)$’ in the base semantical
system, which we now call system $K$. Notice that it is also the case that ‘$Q$’
is semantically entailed by and derivable from ‘$P$’ and ‘$P \supset Q$’ in Sentential
Logic. This suggests that in general where a sentence $\alpha$ is a consequence of a
set of sentences in Sentential Logic, the corresponding necessity-sentence $\Box \alpha$
bears the same relation to the necessity-versions of each sentence in the set in
Modal Sentential Logic.\(^{25}\)

A more general result for $K$ is that $K$-entailment and $K$-derivability are
preserved when the relevant sentences are made into necessity sentences. It is
usually said that necessity is closed under the relations of $K$-entailment and
$K$-derivability.

**Closure of Necessity under $K$-Entailment**

$\{\gamma_1, \cdots, \gamma_n\} \models_{K} \alpha$, then $\{\Box \gamma_1, \cdots, \Box \gamma_n\} \models_{K} \Box \alpha$

**Proof.** Suppose $\{\gamma_1, \cdots, \gamma_n\} \models_{K} \alpha$. Then for any $K$-frame $Fr$, any $I$ based on
$Fr$, and any $w$ in $W$, if $v_I(\gamma_1, w)=T$ and, $\cdots$, and $v_I(\gamma_n, w)=T$, then $v_I(\alpha, w)=T$. Suppose further that $v_I(\Box \gamma_1, w)=T$, and, $\cdots$, and $v_I(\Box \gamma_n, w)=T$.

Then for each $\gamma_i$, $v_I(\gamma_i, w)=T$ for each world accessible to $w$. From the con-
sequence of the original supposition, we have it that $v_I(\alpha, w_i)=T$ for each $w_i$

\(^{24}\)Strictly speaking, there are two base systems of Modal Sentential Logic, one semantical
and one derivational. We shall, for convenience, sometimes speak of a single system $K$
whose derivational rules $KD$ are sound and complete relative to the semantics.

\(^{25}\)Assuming the basic semantical and derivational rules given in this chapter. This will not
hold for systems of modal sentential logic that do not conform to these rules.
accessible to \( w \). Therefore, \( \nu_f(\Box \alpha, w) = T \). Since the result is obtained with arbitrary \( I \) and \( w \) relative to a \( K \)-frame \( Fr \), we have \( \{ \Box \gamma_1, \cdots, \Box \gamma_n \} \models_K \Box \alpha \), which was to be demonstrated.

Corresponding to closure under semantic entailment is closure under derivability.

**Closure of Necessity under \( K \)-Derivability**

\( \{ \gamma_1, \cdots, \gamma_n \} \vdash_{KD} \alpha \), then \( \{ \Box \gamma_1, \cdots, \Box \gamma_n \} \vdash_{KD} \Box \alpha \)

We can demonstrate closure under \( K \)-derivability by providing a schema representing a style of derivatio\( ^{26} \)

\[
\begin{array}{c|c}
\hline
\Box \gamma_1 & \text{Assumption} \\
\vdots & \vdots \\
\Box \gamma_n & \text{Assumption} \\
\hline
\gamma_1 & \text{SR} \\
\vdots & \vdots \\
\gamma_n & \text{SR} \\
\hline
\alpha & \text{SR} \\
\hline
\Box \alpha & \Box \square \end{array}
\]

We first assume each of \( \Box \gamma_1, \cdots, \Box \gamma_n \). Then we introduce a restricted scope line and strictly reiterate each of the assumptions. Given that \( \alpha \) is derivable from \( \{ \gamma_1, \cdots, \gamma_n \} \) with no undischarged assumptions, this derivation can take place entirely within the restricted scope line. And if this is so, then we can end the restricted scope line with the use of \( \Box \) Introduction. Therefore, the derivation may be ended and \( \alpha \) is derivable from \( \{ \gamma_1, \cdots, \gamma_n \} \), which was to be proved.

\( K \)-validity and \( K \)-theoremhood are special cases of semantic entailment and derivability, respectively. They are the cases where \( \{ \gamma_1, \cdots, \gamma_n \} \) is the empty set, \( 0 \models_K \alpha \) and \( 0 \vdash_K \alpha \). So we can obtain corollaries of Closure in the limiting case where \( \{ \gamma_1, \cdots, \gamma_n \} \) is empty. We will call this special case of closure necessitation.

**Necessitation of \( K \)-valid Sentences**

If \( \models_K \alpha \), then \( \models_K \Box \alpha \)

**Proof.** Suppose \( \alpha \) is valid in \( K \). Then \( \alpha \) is true on all worlds on all \( K \)-interpretations. So for any \( K \)-interpretation \( I \), \( \alpha \) is true at all the worlds in

\[26\]The ellipses indicate places where other sentences might occur.
I. From the standpoint of any given world \( w \), \( \alpha \) is true at all accessible worlds in \( I \), in which case \( \Box \alpha \) is true at \( w \). Since this holds for any interpretation, \( \Box \alpha \) is true on all interpretations, which was to be proved.

We can easily give a derivation schema showing that the result holds for \( K \)-derivability.

**Necessitation of \( K \)-Theorems**

If \( \vdash_{KD} \alpha \), then \( \vdash_{KD} \Box \alpha \).

```
\[\begin{array}{c}
\vdash_{KD} \alpha \\
\vdash_{KD} \Box \alpha \\
\end{array}\]
```

If it is assumed that \( \alpha \) is derivable with no undischarged assumptions, then the derivation can take place entirely within a restricted scope line. And if this is so, then we can end the restricted scope line with the use of \( \Box \) Introduction. This can be done within the leftmost restricted scope line, and so the derivation may be ended and \( \Box \alpha \) is a theorem of \( KD \).

Derivational systems (and axiom systems) which have the property of necessitation are called *normal* systems.\(^{27}\)

**Exercise.** Give an example of an entailment and derivation that hold for \( K \) and \( KD \) but not in \( SL \) and \( SD \), respectively. Then show why closure holds in that case.

### 5.2 The Axiom System \( K \)

Most expositions of modal logic present modal systems as axiom systems rather than derivability systems.\(^{28}\) What follows is a very brief introduction to the axiomatic approach, which may be of value to the reader in understanding other work on modal logic.

The goal of axiom systems is to produce a set of theorems, sentences which are either axioms or derived from axioms using rules of inference. As we present it, the axiom system consists of a set of axiom schemata and a set of rules of inference applying to the axioms that conform to the schemata.\(^{29}\) Application of a rule of inference to an axiom or axioms yields theorems, and rules of inference can be applied to the theorems as well.

One can also define a derivability relation for an axiom system, though the emphasis tends to be on the production of theorems. Use of axiom systems to

\(^{27}\)The original axiom system \( S3 \) of C.I. Lewis was not a normal system. Semantics for his systems \( S1, S2 \) and \( S3 \) are not even indirectly truth-functional. Semantically, frames contain “non-normal” worlds at which some non-atomic sentences are assigned truth-values by the valuation function directly. Specifically, \( \Box \alpha \) is always false and \( \Diamond \alpha \) always true.

\(^{28}\)Notable exceptions are Frederick Fitch, *Symbolic Logic*, Kenneth Konyndyk, *Introductory Modal Logic* and Daniel Bonevac, *Deduction*.

\(^{29}\)Axioms are substitution instances of axiom schemata.
develop modal logic (such as $K$) tends to promote the idea that the point of the system is to provide a set of potential truths about the modalities in question, that is, to generate a theory of the modalities. The main approach in this text is to focus on inferences made with modal premises and/or modal conclusions.

The non-modal Sentential Logic axioms are as follows:

$$
\vdash K \alpha \supset (\beta \supset \alpha)
$$

$$
\vdash K (\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\beta \supset \gamma))
$$

$$
\vdash K (\sim \beta \supset \sim \alpha) \supset ((\sim \beta \supset \alpha) \supset \beta)
$$

There is one non-modal rules of inference, Modus Ponens: from $\vdash K \alpha$ and $\vdash K \alpha \supset \beta$, infer $\vdash K \beta$.

The modal component of axiomatic $K$ consists of a single axiom schema:

$$
\vdash K \Box(\alpha \supset \beta) \supset (\Box \alpha \supset \Box \beta)
$$

and a rule of inference, Necessitation:

If $\vdash K \alpha$ then $\vdash K \Box \alpha$

we have already seen that Necessitation is the limiting case of closure.

The $K$ axiom schema is easily shown to be derivable in $KD$.

1. $\Box(\alpha \supset \beta)$ Assumption
2. $\Box \alpha$ Assumption
3. $\alpha$ 2 Strict Reiteration
4. $\alpha \supset \beta$ 1 Strict Reiteration
5. $\beta$ 3 4 $\supset$E
6. $\Box \beta$ 3-5 $\supset$ I
7. $\Box \alpha \supset \Box \beta$ 2-6 $\supset$ I
8. $\Box(\alpha \supset \beta) \supset (\Box \alpha \supset \Box \beta)$ 1-7 $\supset$ I

Casual inspection of this schema for a derivation shows that the reasoning pattern used to justify closure is operative here.\(^{30}\)

5.3 Modal Operators in $K$

5.3.1 Possibility

As was noted in Section 1, necessity and possibility can be defined in terms of each other. If a necessity-sentence is defined in terms of a possibility-sentence, or vice-versa, then the defined sentence is just a notational variant of the original sentence. In that case, we can say that these sentences are definitionally equivalent.

In the semantics, two sentences are equivalent just in case they have the same truth values at all worlds on all interpretations. In the derivational system, two sentences are equivalent just in case each is derivable from the other as an

\(^{30}\)It is the opinion of the author that the derivational system $KD$, aside from being much easier to use than an axiom system, also reveals how modal reasoning actually works. The emphasis in the axiom system on theoremhood and the rule of necessitation obscures the pre-eminent role of closure in system $K$. 

21
assumption, with no other undischarged assumptions. In the semantical and derivational systems, the necessity- and possibility-operators each fall under their own set of rules and are not defined in terms of each other. Nonetheless, we can prove equivalences in $K$ and $KD$ which parallel the definitions that might have been given. These equivalences are commonly known as duality.

Duality

\[ \square \alpha \text{ is equivalent to } \sim \Diamond \sim \alpha \]
\[ \Diamond \alpha \text{ is equivalent to } \sim \square \sim \alpha \]

Derivatively, there are two other equivalences that are also commonly brought under the heading of Duality.

\[ \square \sim \alpha \text{ is equivalent to } \sim \Diamond \sim \alpha \]
\[ \Diamond \sim \alpha \text{ is equivalent to } \sim \square \sim \alpha \]

For the semantical system, we use the truth-definitions for the ‘\(\square\)’ and the ‘\(\Diamond\)’ to prove the equivalence. For the derivational system, the rules of inference for the two operators are used to complete the proof. We will prove the first of the two semantic equivalences and leave the second as an exercise.

On an arbitrary interpretation $I$ and arbitrary world $w$ in $I$,

\[ v_I(\square \alpha, w) = T \text{ iff } \]
\[ \text{for all } w_i \text{ such that } Rww_i, v_I(\alpha, w_i) = T \text{ iff } \]
\[ \text{for all } w_i \text{ such that } Rww_i, v_I(\sim \alpha, w_i) = F \text{ iff } \]
\[ v_I(\Diamond \sim \alpha, w) = F \text{ iff } \]
\[ v_I(\sim \Diamond \sim \alpha, w) = T \text{.} \]

Since the choice of interpretations and worlds is arbitrary, $\square \alpha$ and $\sim \Diamond \sim \alpha$ have the same truth-value on all interpretations, and so they are semantically equivalent.

**Exercise** Show that the second part of duality holds.

To show that the two sentences are inter-derivable, we must give two separate derivations, using both the rules for the ‘\(\square\)’ and the ‘\(\Diamond\)’. The derivations proceed in a straightforward manner because the Impossibility rules were designed to accommodate Duality.

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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>$\square \alpha$</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>$\alpha$</td>
<td>1 SR</td>
</tr>
<tr>
<td>3</td>
<td>$\sim \sim \alpha$</td>
<td>2 DN</td>
</tr>
<tr>
<td>4</td>
<td>$\sim \Diamond \sim \alpha$</td>
<td>3 $\sim \Diamond$ I</td>
</tr>
<tr>
<td>1</td>
<td>$\sim \Diamond \sim \alpha$</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>$\sim \sim \alpha$</td>
<td>1 SR($\sim \Diamond$)</td>
</tr>
<tr>
<td>3</td>
<td>$\alpha$</td>
<td>2 DN</td>
</tr>
<tr>
<td>4</td>
<td>$\square \alpha$</td>
<td>2-3 $\square$ I</td>
</tr>
</tbody>
</table>
Exercise. Show that $\{\sim \Box \sim \alpha\} \vdash K \lozenge \alpha$.

The reader might suspect that some question has been begged in using the Impossibility rules to prove Duality, since those rules have their origin in the definitional versions of Duality given in Section 1. But the definitions have not been presupposed to prove duality. The Impossibility rules are motivated by Duality definitions but do not depend on the definitions in any formal way. Given the Impossibility rules, if we define the ‘$\lozenge$’ in terms of the ‘$\Box$’, then the resulting derivations yield the definitional equivalents of the derivations using the ‘$\Box$’ rules. What has been shown here is that given both sets of rules, we can demonstrate the equivalences given in the definitions without presupposing the definitions. Note also that the Impossibility rules were shown to be sound relative to the semantics for the ‘$\lozenge$’ in $K$.

5.3.2 Strict Implication

We saw in Section 3 that necessity-sentences can be understood semantically as mirroring in the object-language the meta-logical property of validity. The same kind of mirroring relation holds between a “strict implication” $\alpha \prec \beta$ and the meta-logical relation of semantic entailment. The original modal axiom systems of C. I. Lewis were centered around strict implication. In a later book, Lewis contrasted the way modal logic treats deduction with the way it is treated in the “truth-value” system $SL$.

The principles of Strict Implication express the facts about any such deduction in an explicit manner in which they cannot be expressed within the truth-value system itself, for the reason that, in Strict Implication, what is tautological is distinguished from what is merely true, whereas this difference does not ordinarily appear in the symbols of the truth-value system. (Symbolic Logic (1932), 247)

The “tautological” is expressed by the necessity-sentence in modal logic but cannot be expressed in non-modal Sentential Logic.

Lewis wanted to treat formally the relation of semantic entailment. This much is clear from his early discussion of the meaning of strict implication.

The strict implication, $p \prec q$, means, “It is impossible that $p$ be true and $q$ be false,” or “$p$ is inconsistent with the denial of $q$.” Any set of mutually consistent propositions may be said to define a “possible situation” or “case” or “state of affairs”. And a proposition may be “true” of more than one such possible situation—may belong to more than one such set... In these terms, we can translate $p \prec q$ by “Any situation in which $p$ should be true and $q$ false is impossible”.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Assumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\lozenge \alpha$</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>$\Box \sim \alpha$</td>
<td>Assumption</td>
</tr>
<tr>
<td>3</td>
<td>$\parallel \sim \alpha$</td>
<td>2 SR</td>
</tr>
<tr>
<td>4</td>
<td>$\sim \lozenge \alpha$</td>
<td>3 $\sim \lozenge$ I</td>
</tr>
<tr>
<td>5</td>
<td>$\sim \Box \sim \alpha$</td>
<td>2-4 $\sim$ I</td>
</tr>
</tbody>
</table>
From the standpoint of our formal semantics, we are interested in the case of semantic entailment where the entailing set of sentences contains only one member:

$$\{\gamma\} \vDash_K \alpha.$$  

For example, we know that

$$\{P \& Q\} \vDash_K P.$$  

This entailment holds because at any world \(w\) on any interpretation \(I\), if \(P \& Q\) is true at \(w\) then \(Q\) is true there as well.

We can use this idea to give a truth-definition for sentences of the form \(\alpha \prec \beta\). We will say that a sentence of this form is true at a world just in case at all accessible worlds, if the antecedent is true, then the consequent is true.

**Truth-Definition for Strict Implication**

\[v_I(\alpha \prec \beta, w) = T \iff \text{at all worlds } w_i \text{ in } I \text{ such that } Rww_i, \text{ if } v_I(\alpha, w_i) = T, \text{ then } v_I(\beta, w_i) = T.\]

The difference between the truth-definition for the ‘\(\prec\)’ and the definition of semantic entailment is that the truth-definition gives a strict implication-sentence a truth-value only on a single interpretation.

With this truth-definition in hand, we can give a derivational rule for strict implication. This rule will build on the notion of a Modal Assumption used in the rule for \(\Box\) Elimination. The idea is that if we make a Modal Assumption of \(\alpha\) and derive \(\beta\) from \(\alpha\), then we can close the hypothetical restricted scope line and get \(\alpha \prec \beta\). This follows the truth-definition exactly: if we suppose an accessible world at which the truth of \(\beta\) follows from that of \(\alpha\), then we can say that if \(\alpha\) is true there, \(\beta\) is as well.

**\(\prec\) Introduction**

\[
\begin{array}{c}
\alpha \\
\cdot \\
\cdot \\
\beta \\
\hline
\alpha \prec \beta 
\end{array}
\]

As an example of the use of this rule, we will prove a well-known result that will be discussed in a later section: \(\vdash_K \Box P \prec (Q \prec P)\).

\[
\begin{array}{c|l}
1 & \Box P & \text{Modal Assumption} \\
2 & P & \text{1 SR} \\
3 & Q \prec P & \text{2-3 } \prec I \\
4 & \Box P \prec (Q \prec P) & \text{1-4 } \prec I \\
\end{array}
\]

This result, that a necessary sentence is implied by any sentence, is known as
one of the “paradoxes of strict implication”, which will be discussed in detail later.

Inspection of the derivational rule reveals that if \( \{\alpha\} \vdash K \beta \) then \( \vDash K \alpha \prec \beta \).

Any theorem can be derived with a restricted scope line. The same result holds semantically: if \( \{\alpha\} \models K \beta \), then \( \vDash K \alpha \prec \beta \).

**Proof.** If \( \{\alpha\} \models K \beta \), then for all worlds \( w \) in all interpretations \( I \), if \( v_I(\alpha, w_i) = T \), then \( v_I(\beta, w_i) = T \). Therefore, if a world \( w_i \) is accessible to a given world \( w \), if \( v_I(\alpha, w_i) = T \), then \( v_I(\beta, w_i) = T \). So for all worlds \( w \) and all interpretations \( I \), \( v_I(\alpha \prec \beta, w) = T \), which is to say that \( \vDash K \alpha \prec \beta \).

This result establishes that strict implication mirrors in the object-language the meta-logical relation of semantical entailment. It is important to note that what counts as a semantical entailment is different relative to different semantical systems. Exactly which set of valid strict implications is produced by a semantical system is vital to its adequacy as a logic of “implies”.

Strict implication can be understood as a necessitated material conditional. \( \alpha \prec \beta \) is semantically equivalent to \( \Box(\alpha \supset \beta) \). Where \( Rww_i \), given that if \( v_I(\alpha, w_i) = T \), then \( v_I(\beta, w_i) = T \), it follows from the truth-definition for the ‘\( \supset \)’ that \( v_I(\alpha \supset \beta, w_i) = T \). And this is the condition needed for the truth of \( \Box(\alpha \supset \beta) \) at \( w \). The converse holds by the same reasoning.

The same result holds in the derivational system. If \( \alpha \supset \beta \) is derivable from no assumptions in a restricted scope line, we can end the scope line and affix a ‘\( \Box \)’ to the front of it. And this can be done just in case we can derive \( \beta \) from the Modal Assumption \( \alpha \) and no other assumptions, which entitles us to end the restricted scope line and write \( \alpha \prec \beta \).

With this information, we can give a rule for strictly reiterating sentences whose main operator is the ‘\( \prec \)’.

**Strict Reiteration for Strict Implication**

\[
\begin{array}{c|c}
\alpha \prec \beta & \alpha \supset \beta \quad \text{SR}(\prec) \\
\hline
\alpha & \alpha \supset \beta \quad \text{SR}(\prec) \\
\hline
\end{array}
\]

Provided that only one restricted scope line is crossed

**Exercise.** Justify this rule as a derived rule of \( KD \).

Using this result, we can show that necessity and possibility of a single sentence are closed under strict implication.
Exercise. Give a derivation-schema that shows closure of a single possibility-sentence over strict implication.

The closure results show once again how strict implication precisely mirrors the consequence relation in $K$ and $KD$.

5.3.3 Other Modal Operators

Strict implication can be understood as the result of prefixing a modal operator to (or “modalizing”) Sentential Logic sentences of the form $\alpha \supset \beta$ to get $\Box(\alpha \supset \beta)$. Other SL sentence-forms are subject to the same procedure. A modalized SL form can serve as the basis for defining other modal operators that have appeared in the literature of modal logic. Here is a list of all the possibilities.

- $\Box \neg \alpha$
- $\Diamond \neg \alpha$
- $\Box(\alpha \& \beta)$
- $\Diamond(\alpha \& \beta)$
- $\Box(\alpha \lor \beta)$
- $\Diamond(\alpha \lor \beta)$
- $\Box(\alpha \supset \beta)$
- $\Diamond(\alpha \supset \beta)$
- $\Box(\alpha \equiv \beta)$
- $\Diamond(\alpha \equiv \beta)$

Three of these are of historical interest and will be examined here: $\Box(\alpha \equiv \beta)$, $\Box(\alpha \lor \beta)$, and $\Diamond(\alpha \& \beta)$.

In Lewis’s systems, strict equivalence is the conjunction of two strict implications, $(\alpha \prec \beta) \land (\beta \prec \alpha)$. A sentence of this form is true at a world on an interpretation just in case $\alpha$ and $\beta$ have the same truth-values at all accessible worlds. This can be seen from the fact that if $\alpha$ is true at an accessible world, $\beta$ is true there as well, by the truth-definition of the ‘$\prec$’. By the same reasoning, if $\beta$ is true at an accessible world, then $\alpha$ is also true there. If $\alpha$ is false at an accessible world, then $\beta$ is false there. And if $\beta$ is false at an accessible world, then so is $\alpha$.

Lewis explained strict equivalence as a relation that “denotes an equivalence of logical import or meaning, while $p \equiv q$ denotes simply an equivalence of truth-value” (Survey of Symbolic Logic, p. 294). He could have added that a strict equivalence between $\alpha$ and $\beta$ can be understood as the result of prefixing a necessity operator to a biconditional: $\Box(\alpha \equiv \beta)$. In our semantics, a sentence of this form is true at a world just in case the embedded biconditional is true at all accessible worlds. That is, at each accessible world, either $\alpha$ and $\beta$ are both true there or they are both false there. As can be seen, this is exactly what makes a strict equivalence true.

Exercise. Give a derviation to prove the following: $\{ (\alpha \prec \beta) \land (\beta \prec \alpha) \} \vdash_K \Box(\alpha \equiv \beta)$.

The same result can be obtained by appeal to closure. The conjunction $(\alpha \prec \beta) \land (\beta \prec \alpha)$ is equivalent to $\Box(\alpha \supset \beta) \land \Box(\beta \supset \alpha)$. We know from SL that

31: Prefixing a ‘$\Box$’ to a negated sentence yields impossibility, which has already been treated above.

32: In contemporary work on modal logic, sometimes a double fish-hook ‘$\succ \prec$’; and sometimes a quadruple bar, is used to indicate strict equivalence. Lewis used the identity sign ‘$=$’.
\(\alpha \supset \beta\) and \(\beta \supset \alpha\) are both consequences of \((\alpha \equiv \beta)\). By closure, \(\Box(\alpha \supset \beta)\) and \(\Box(\beta \supset \alpha)\) are both consequences of \(\Box(\alpha \equiv \beta)\). And \(\Box(\alpha \supset \beta) \& \Box(\beta \supset \alpha)\) is a consequence of these two sentence forms. Conversely, by closure, \(\Box(\alpha \equiv \beta)\) is a consequence of \(\Box(\alpha \supset \beta)\) and \(\Box(\beta \supset \alpha)\). And these two sentences are consequences of \(\Box(\alpha \supset \beta) \& \Box(\beta \supset \alpha)\). So \(\Box(\alpha \equiv \beta)\) is a consequence of \(\Box(\alpha \supset \beta)\) and \(\Box(\beta \supset \alpha)\). And these two sentences are consequences of \(\Box(\alpha \supset \beta) \& \Box(\beta \supset \alpha)\). So \(\Box(\alpha \equiv \beta)\) and \(\Box(\alpha \supset \beta) \& \Box(\beta \supset \alpha)\) are consequences of each other. That is, they are equivalent forms.

Note also that single possibility- and necessity-sentences are closed under strict equivalence. This follows from the fact that strict equivalence-sentences are conjunctions of strict implication sentences, for which closure holds.

Another modal operator that can be formed by modalizing an SL sentence-form is intensional disjunction: \(\Box(\alpha \lor \beta)\). Lewis started his investigations into modal logic with a discussion of this operator (though without symbolizing it).\(^{33}\)

In *A Survey of Symbolic Logic*, he used a ‘\(\lor\)’ for intensional disjunction and a ‘\('+\)’ for truth-functional disjunction. Here is how he described the difference.

Both \(p \lor q\) and \(p + q\) would be read as “Either \(p\) or \(q\)”. But \(p \lor q\) denotes a necessary connection; \(p + q\) a merely factual one. Let \(p\) represent “Today is Monday”, and \(q\), “2 + 2 = 4”. Then \(p + q\) is true but \(p \lor q\) is false. In point of fact, at least one of the two propositions, “Today is Monday”, and “2 + 2 = 4”, is true; but there is no necessary connection between them. “Either … or …” is ambiguous in this respect. Ask the members of any company whether the proposition “Either today is Monday or 2 + 2 = 4” is true, and they will disagree. Some will combine “Either … or …” to the \(p \lor q\) meaning, others will make it include the \(p + q\) meaning; few, or none, will make the necessary distinction. (*A Survey of Symbolic Logic*, p. 294)

An example of the kind of sentence Lewis had in mind as a necessary disjunction is (in the notation of this text) \(\alpha \lor \sim \alpha\). Sentences with this form are valid in \(K\), and by necessitation, \(\Box(\alpha \lor \sim \alpha)\) is valid in \(K\) as well.

We have already introduced a symbol for consistency, which produces sentences of the form \(\alpha \circ \beta\). We can lay down the following truth-condition for consistency:

**Truth-Definition for Consistency**

\[
v_I(\alpha \circ \beta, w) = T \text{ if there is a world } w_i \text{ in } I \text{ such that } R_{ww_i} \text{ and both } v_I(\alpha, w_i) = T \text{ and } v_I(\beta, w_i) = T.
\]

Given this semantical rule, sentences of this form are equivalent to \(\Diamond(\alpha \& \beta)\). With this equivalence in mind, derivational rules for the ‘\(\circ\)’ could easily be devised, but as they are not of any intrinsic interest, they will not be given here.

Lewis showed a number of properties of consistency (in his system \(S3\)), including these: “If \(p\) and \(q\) are consistent, then \(q\) and \(p\) are consistent”, “If \(p\) and \(q\) are consistent, then it is possible that \(p\) be true”, “If it is possible that \(p\) be true”, “If it is possible that \(p\)
be true, then \( p \) is consistent with itself ("A Survey of Symbolic Logic", p. 300). Viewed as consequence-relations or strict implications, these hold in \( K \) as well.

**Exercise.** Symbolize these conditionals as strict implications. Show that they hold in the semantics for \( K \).

Of some interest is an entailment of \( S3 \) (and \( KD \)) discussed by Lewis:

\[(\alpha \circ (\beta \& \gamma)) \vDash_K (\beta \circ (\alpha \& \gamma)).\]

We can see that this holds because it is equivalent to

\[\lozenge(\alpha \& (\beta \& \gamma)) \vDash_K \lozenge(\beta \& (\alpha \& \gamma)).\]

The non-modal conjunctions inside the possibility-operators are clearly equivalent, as can be seen from the Sentential Logic derivational rule of Association. But the consistency relation itself is not associative in \( K \).

\[{\lozenge(\alpha \& \lozenge(\beta \& \gamma))} \vDash_K {\lozenge(\beta \& \lozenge(\alpha \& \gamma))}\]

This was recognized by Lewis, who remarked that because of the failure, the treatment of consistency "seems incomplete" ("Survey", p. 300). The reason for the failure can be seen quite readily when an equivalent form is displayed.

\[{\lozenge(\lozenge(\alpha \& \lozenge(\beta \& \gamma)))} \vDash_K {\lozenge(\lozenge(\beta \& \alpha \& \gamma))}\]

Suppose that we have an interpretation \( I \) with worlds \( w, w_1, \) and \( w_2 \), and that \( Rw_w \) and \( Rw_{w_1} \). But suppose further that it is not the case that \( Rw_{w_2} \) or that any other accessibility relation holds. Let \( \alpha \) be true and \( \gamma \) false at \( w_1 \). At \( w_2 \), let \( \beta \) and \( \gamma \) both be true. In that case, \( \beta \& \gamma \) is true at \( w_2 \), and so \( \lozenge(\beta \& \gamma) \) is true at \( w_1 \), in which case, so is \( \alpha \& \lozenge(\beta \& \gamma) \). Then \( \lozenge(\alpha \& \lozenge(\beta \& \gamma)) \) is true at \( w \). Now let \( \gamma \) be false at \( w_1 \), in which case \( \lozenge(\beta \& \alpha \& \gamma) \) is false at \( w_1 \). Since \( w_1 \) is the only world accessible to \( w \), \( \lozenge(\lozenge(\beta \& \alpha \& \gamma)) \) is false there as well, which was to be demonstrated. Unlike the other results, this failure is not overcome in the stronger systems we will be considering. No restrictions on accessibility will block the kind of counter-example just given. Non-associativity seems to be a fundamental fact about consistency.

### 6 Non-Consequences in \( K \)

The semantical system \( K \) and its derivational twin \( KD \) are very weak. A number of consequence relations which one might expect to hold, given an intended informal interpretation of the logic, do not. These failures mean that the system is inadequate for those intended uses of modal logic. Before turning to these applications, we will look at some failures of consequence that would seem to present a problem for most applications.

Because a \( K \)-frame may have any relation of accessibility holding of its worlds, some \( K \)-frames will contain "dead-ends", worlds to which no world is accessible. On any interpretation based on such a frame \( Fr \), a sentence of the form \( \lozenge \alpha \) will be false at such a dead-end. Therefore,
And since such a frame $Fr$ is a member of the class of all $K$-frames, we may assert:

$$\not\vDash_{Fr} \Diamond \alpha.$$  

Because of soundness, the corresponding failure occurs in the derivational system. An examination of the rules shows that there is no rule for the introduction of a possibility operator. Adding such a rule will allow theorems of the form $\Diamond \alpha$. The failure is overcome in system $D$, to be discussed in the next chapter.

A related failure is that a necessity-sentence $\square \alpha$ does not have as a consequence the possibility-sentence $\Diamond \alpha$.

$$\square \alpha \not\vDash_{K} \Diamond \alpha.$$  

The reason is that when a necessity sentence holds at a dead-end, the possibility sentence does not. More formally, consider a frame with a world $w$ such that there is no $w_i$ such that $Rww_i$. Assume that $v_I(\square \alpha, w) = T$. Because there are no worlds accessible to $w$, $v_I(\Diamond \alpha, w) = F$.

Another significant non-consequence is this:

$$\square \alpha \not\vDash_{K} \alpha.$$  

A necessity-sentence does not have its embedded sentence as a consequence. The reason here is that in some frames, there are worlds that are not accessible to themselves. This allows an interpretation $I$ on whose valuation function the embedded sentence $\alpha$ fails to hold at the world where it is evaluated but does hold at all accessible worlds. The accessibility relation need not be reflexive on a $K$-frame. Thus we can construct a frame with a world $w$ such that it is not the case that $Rww$ and on which $v_I(\square \alpha, w) = T$. Because there are no worlds accessible to $w$, $v_I(\Diamond \alpha, w) = F$.

The derivational system for $T$ will include a rule of $\square$ Elimination which will directly yield the consequence in question.

Detachment for strict implication also fails in $K$.

$$\{\alpha, \alpha \prec \beta\} \not\vDash_{K} \beta.$$  

This is most readily seen in the derivational system. If a sentence $\alpha$ occurs on a line of a derivation, and $\alpha \prec \beta$ occurs on another line, one may not infer $\beta$. One would have to initiate a hypothetical restricted scope line, but $\alpha$ may not have a form which would allow the use of Strict Reiteration across that line. From a semantical standpoint, the reason is that there may be an assignment of $T$ to $\alpha$ and $F$ to $\beta$ at a world $w$, despite the fact that $\alpha \prec \beta$ is assigned $T$ there. This can occur if $w$ is not accessible to itself. While at all accessible worlds where $\alpha$ is true, $\beta$ must be assigned $T$, $w$ is not one of those worlds, and so it would not be possible for $\beta$ to hold there.

---

34 A slash through a turnstile or double-turnstile indicates that the relation indicated by the symbol does not hold.

35 In what follows, we shall give the semantical result by itself, with the understanding that because of soundness, the corresponding result holds for the derivational system.
is permissible that \( \beta \) be assigned \( F \) there. System \( T \) will yield detachment for strict implication.

It might be thought that what is necessary is necessarily necessary, but

\[
\{\Box \alpha\} \not\models_K \Box \Box \alpha.
\]

This failure is due to the fact that the accessibility relation on a \( K \)-frame does not have to be transitive. Based on frames where \( Rww_1 \) and \( Rww_2 \), but not \( Rww_2 \), interpretations \( I \) can be given on which \( v_I(\Box P, w) = T \) and \( v_I(\Box \Box P, w) = F \). Consider a frame on which \( Rww_1 \) and \( Rww_2 \), but nothing else, and an interpretation \( I \) based that frame. Let \( v_I(P, w_1) = T \) and \( v_I(P, w_2) = F \).

Since \( w_1 \) is the only world accessible to \( w \), \( v_I(\Box P, w) = T \). But \( v_I(\Box \Box P, w_1) = F \), since \( v_I(P, w_2) = F \) at the only accessible world. So \( v_I(\Box \Box P, w) = F \), which was to be proved. In the next chapter, we shall see how system \( S' \) allows this consequence. The derivational system for \( S' \) will include a more liberal rule of Strict Reiteration.

An even stronger thesis is that what is possible is necessarily possible:

\[
\{\diamond \alpha\} \not\models_K \Box \diamond \alpha.
\]

Here the result follows from the fact that a \( K \)-frame need not be euclidean. A relation is euclidean when, if \( Rww_1 \) and \( Rww_2 \), then \( Rww_1 \). So consider a frame in which \( Rww_1 \) and \( Rww_2 \), and nothing else. Let \( v_I(\Box P, w_1) = T \). Then \( v_I(\Box P, w_2) = T \). Now let \( v_I(\Box P, w_2) = F \). Then \( v_I(\Box \Box P, w_1) = F \). So \( v_I(\Box \Box P, w) = F \), which was to be proved. System \( B \) overcomes this counter-example. As with \( S' \), a more liberal Strict Reiteration rule will be added to get a derivational system for \( B \) to allow the consequence.

In the general case, closure of possibility over the consequence relation does not hold in \( K \) or any stronger system based on it. That is,

\[
\{\diamond \alpha\} \not\models_K \Box \diamond \alpha.
\]

It is not the case that if \( \{\gamma_1, \ldots, \gamma_n\} \models_K \alpha \), then \( \{\diamond \gamma_1, \ldots, \diamond \gamma_n\} \not\models_K \diamond \alpha \).

**Proof.** It is enough to show that the consequence does not hold in a single instance, \( \{\diamond P, \diamond Q\} \not\models_K \diamond (P \land Q) \), where \( \{P, Q\} \models_K P \land Q \). Consider a \( K \)-frame \( Fr = <W, R> \) where \( W = \{w, w_1, w_2\} \) and \( Rww_1, Rww_2 \). Let \( I \) be based on \( Fr \), with the following assignments: \( v_I(P, w_1) = T \), \( v_I(Q, w_1) = F \), \( v_I(P, w_2) = F \), and \( v_I(Q, w_1) = T \). Now \( v_I(\diamond P, w) = T \) and \( v_I(\diamond Q, w) = T \), by the truth-definition for ‘\( \diamond \)’. However, it is also the case that \( v_I(P \land Q, w_1) = F \) and \( v_I(P \land Q, w_2) = F \). Therefore, \( v_I(\diamond (P \land Q), w) = F \). No standard restriction on accessibility will prevent a counter-example to be formulated, since nothing forces the assignment of specific truth-values to sentence-letters at a given world.

## 7 Applications of \( K \)

Now that system \( K \) has been thoroughly investigated, we are in a position to discuss its adequacy for representing various modalities that we use in our reasoning. Such representations will be called *applications* of the semantical and derivational systems. In this section, we shall examine \( K \) as a logic of truth
(alethic modal logic), implication (implicative logic), obligation (deontic logic), belief (doxastic logic), knowledge (epistemic logic), and time (temporal logic). The general result will be that $K$ is a reasonable basis for such systems, though it generally is inadequate to represent these relations without being strengthened.

### 7.1 Alethic Modal Logic

It is natural to think of $K$ as a formal logic of truth. In the formal semantics, we interpret its sentences as having one of two values, $T$ or $F$, but not both. In principle, these values could be anything from numbers (e.g. the binary numbers 0 and 1), to two people, to truth and falsehood. And so the semantics for $K$ could also be understood as yielding a logic of switching-circuits, for example, where the values given to sentences reflect open or closed switches on a circuit. Or it could be interpreted as embodying principles of Boolean algebra. It is of historical interest to note that C. I. Lewis traced what he regarded as the deficiencies of $SL$ to its origins in Boolean algebra.

If we want to understand modal logic alethically, it is best to understand the non-modal basis of the system, $SL$ alethically as well. We can understand $SL$ as a minimal basis for a logic of truth. The semantics for $SL$ is intended to capture generally-shared beliefs about the nature of truth and falsehood. The requirement that every sentence be given one of only two values is widely (though not universally) held to apply to truth and falsehood. The requirement that no sentence have more than one value reflects a view held even more widely (though again, not universally), that whatever is true is not false, and vice-versa.

Moreover, the truth-definitions for non-modal sentences embody other aspects of what is true or false, e.g., that a sentence is true if and only if its negation is false. It should be noted, however, that the convenient truth-definition for the ‘$\supset$’ does not capture a good deal of the force of conditional sentences. So we can understand the semantics for $SL$ as yielding a logic of truth, but it may be one that is too permissive. Lewis held this view, and it spurred him to develop modern modal logic.

Now we can say that modal logic allows us to express in a formal way some of the properties of the underlying alethic logic. So, we take $T$ and $F$ to be truth and falsehood, and we understand the ‘□’ to indicate necessary truth and the ‘♦’ to indicate possible truth. In this way, we provide an enriched logic of truth that can make distinctions, within its formal language, that cannot be made in $SL$. For example, it allows a representation of “strict implication” that may reflect better the way we understand some conditional sentences of English, and the representation of “intensional disjunction” that may satisfy those who think that “Either today is Monday or $2 + 2 = 4$” is not a true sentence.

But once we have arrived at this point, things become somewhat murky. We have a more expressive language which has been given a formal semantics that might be used to give us a logic of necessary and possible truth. We want to know, of course, whether the semantics is adequate for this purpose. But then we face the questions: What is necessary truth? What is possible truth? Historically, different answers have been given to these question, and it turns

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out that no single system of modal logic is adequate to capture all the different ways of understanding necessary and possible truth. This consideration helps motivate the study of a number of different modal systems. Here is an excellent description of the situation in Hughes and Cresswell’s 1968 *An Introduction to Modal Logic*.

This multiplicity of systems is apt to provoke the question, Which system is the correct one? Now the assumption behind this question seems to be that we have in mind some single sense of ‘necessity’ and ‘possibility’, and that systems weaker than the correct one will give us less than the whole truth, while stronger systems will contain theses which even if plausible are really false. But perhaps the systems are not rivals in this way. It is at least possible that a number of systems may each give us the truth about necessity and possibility, though each in a somewhat different sense of those terms. Merely constructing semantic models will not by itself give us an adequate characterization of these different senses; for that a great deal of intricate philosophical work would be required, though, as we have seen the semantic models can give us valuable help in this task. (p. 79)

Our interest is to get some idea of whether the systems presented here are good formalizations of necessity and possibility in different senses of those terms. In what follows, we will do a very small portion of the intricate philosophical work required.

We begin with a distinction acknowledged by nearly everyone who has engaged in the philosophical investigation of necessity and possibility. This is the distinction between *absolute* and *hypothetical* (or *relative*) necessity and possibility. These distinctions can be traced back to Aristotle, who invented modal logic. They were put to extensive use in the hands of Leibniz. And it will be seen that the notion of hypothetical necessity and possibility fits well with the possible-worlds semantics developed in Section 3 above.

For Leibniz, “a truth is **necessary** when its opposite implies a contradiction” (Letter to Coste, December 19, 1707). We may call this kind of necessity “absolute” in the sense that the only non-logical information relevant to the necessity of the truth is found in the truth itself. (Of course, if we are presented with a plethora of accounts of implication, this notion is not so absolute as Leibniz believed.) A truth is hypothetically necessary when it is “based on the hypothesis of the choice made” (*ibid*). Here, Leibniz had in mind the choice God made in creating this, rather than some other, possible universe.

In the semantics for *K*, we can express the Leibnizian notion of absolute necessity in this way:

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36 By “models”, they mean what we have been calling “interpretations”.
37 See William and Mary Kneale, *The Development of Logic*, pp. 92ff.
38 It may be that Leibniz’s account of absolute necessity and possibility is unsatisfactory, though it is not clear what might take its place.
\( \alpha \) is necessarily true iff for some \( \beta \), \( \sim \alpha \models K \beta \& \sim \beta \).

Does the semantics for \( K \) reflect this condition? The question is complicated by the fact that a sentence of the form \( \Box \alpha \) is given a value only at a world on an interpretation. Suppose we understand the necessary truth of \( \alpha \) as necessary truth at a world in an interpretation, without specifying anything further about the world and the interpretation:

\( \alpha \) is necessarily true iff \( v_I(\Box \alpha, w) = T \).

Can we then say that, \( v_I(\Box \alpha, w) = T \) iff for some \( \beta \), \( \sim \alpha \models K \beta \& \sim \beta \)?

We can go from the more general right-hand side of the biconditional to the indefinite left-hand side. If we assume that for some \( \beta \), \( \sim \alpha \models K \beta \& \sim \beta \), this means that at any world on any interpretation that gives \( \sim \alpha \) the value \( T \), \( \beta \& \sim \beta \) also gets the value \( T \) for some \( \beta \). But there is no such world on any interpretation, because this would require that \( \beta \) be given both the value \( T \) and the value \( F \), which is not permitted by the semantics. Therefore, there is no world on any interpretation at which the interpretation gives \( \sim \alpha \) the value \( T \), in which case there is none which gives \( \alpha \) the value \( F \), and so all interpretations give \( \alpha \) the value \( T \) at all worlds. The key here is that the right-hand side is satisfied because of consequences of the left-hand side for all worlds in all interpretations.

If we try to make an argument in the other direction, we get stuck. If we assume that \( \Box \alpha \) gets the value \( T \) at some unspecified world \( w \) in an unspecified interpretation \( I \), all that we can get is that \( \alpha \) gets the value \( T \) at all worlds accessible to \( w \). To be sure, we have the result that \( \sim \alpha \) gets the value \( F \) at all such worlds, in which case, trivially, if \( \alpha \) gets the value \( T \) there, then so does \( \beta \& \sim \beta \). But this will not get us the entailment we seek, because the result is restricted to accessible worlds. This illustrates how basing the semantics on the accessibility relation and relativizing truth to worlds can be inhospitable to the notion of absolute necessity.

Turning to absolute possibility, if we take Leibniz’s account of absolute necessity as a starting-point, we might wish to say that what is absolutely possible is what does not entail a contradiction.

\( \alpha \) is possibly true iff for no \( \beta \), \( \alpha \models K \beta \& \sim \beta \).

As before, we cannot, in the semantics for \( K \), establish the following equivalence for an indefinite \( w \) and \( I \),

\( v_I(\Diamond \alpha, w) = T \) iff for no \( \beta \), \( \alpha \models K \beta \& \sim \beta \).

The result can be established from left to right. If \( v_I(\Diamond \alpha, w) = T \), then for some world \( w_1 \) accessible to \( w \), \( \alpha \) gets the value \( T \). And so at \( w_1 \), \( v_I(\Diamond \alpha, w) = T \) and \( v_I(\beta \& \sim \beta, w_1) = F \). So it is not the case that \( \alpha \models K \beta \& \sim \beta \) for arbitrary \( \beta \) and hence for any \( \beta \).

The converse runs into counter-examples. Even if a sentence does not entail any contradiction, it may have the value \( F \) at all worlds accessible to \( w \), and
so $\Diamond \alpha$ will get the value $F$ at $w$. This will be the case at any dead-end, for example.

Another criticism of the use of $K$-semantics to represent these absolute modalities is that the determination of the truth-value of a modal sentence is relativized to accessible worlds. Leibniz himself is credited with the leading idea of possible-worlds semantics, that a sentence is necessarily true if and only if it is true at all possible worlds. One version of the semantics for the stronger system $S5$ is based directly on this idea, and it might be thought better suited to the idea of absolute possibility and necessity. Then we can say that a sentence of the form $\Diamond \alpha$ is true, period, just in case $\alpha$ is true at a possible world, period.

Kenneth Konyndyk, for example, claims that “broadly logical necessity”, which would seem to be a form of absolute necessity, is best captured by $S5$.

That which is broadly logically necessary is true in all possible worlds, true no matter what. When we use this picture of possible worlds and accessibility relationships, the necessity of a proposition in a given world—in the actual world, for example—is the truth of that proposition in all possible worlds accessible to the actual world. But this would not coincide with truth in all possible worlds unless the accessibility relation were reflexive, symmetric, and transitive. So if we think of broadly logically necessary truths as true in all possible worlds, $S5$ seems to be the most adequate modal system. *Introductory Modal Logic*, p. 60.

It might be responded that we can represent some features of broadly logical necessity in the semantics for $K$. For example, all valid sentences of Sentential Logic are such that necessity-sentences formed from them are $K$-valid.

Still, one might think that this does not go far enough. On an adequate representation of absolute necessity, it can be argued is that $\{\Box \alpha\}$ should entail $\Diamond \alpha$, which does not hold in $K$. Any sentence whose negation entails a contradiction should be thought to be true at some world or other. Moreover, it seems that $\{\Box \alpha\}$ should entail $\alpha$ as well. Leibniz, for example, held that God’s non-existence implies a contradiction, and he concluded from this that God exists. We shall not comment here on whether his premise is correct, but it seems reasonable to draw the conclusion Leibniz does from that premise.

Another apparent requirement for absolute necessity that is not met by $K$ is that $\{\Box \alpha\}$ should entail $\Box \Box \alpha$. The idea is that if $\alpha$ is absolutely necessary, then it is contradictory to deny its necessity, and so, it is necessarily necessary. There is no such entailment in $K$, however, because $\Box \alpha$, if it has the value $T$, only has that truth-value at a world $w$, and as seen in the last section, the accessibility relation may allow $\Box \Box \alpha$ to have the value $F$ at $w$. On the other hand, if $\Box \alpha$ is $K$-valid, then $\{\Box \alpha\}$ does entail $\Box \Box \alpha$. So the semantics for $K$ goes some way toward satisfying this apparent requirement for absolute possibility.

Now let us turn to hypothetical modalities. The semantics for $K$ seems well-suited as a basis for representing them, given that the truth of necessity- and possibility-sentences is defined relative to accessible worlds. As mentioned in
Section 3, accessibility dictates the truth-values that “matter” in determining whether a sentence is necessarily or possibly true. The accessibility relation itself might best be understood as defining which possibilities matter under a condition that is taken to hold. The condition itself is never expressed in the semantics. It is instead implicit in the arrangement of worlds under the accessibility relation.

Is $K$ an adequate system to express hypothetical necessity and possibility? The fundamental strength of $K$ is the property of closure of necessity and possibility over the consequence relation. Basically, closure reflects the fact that truth-values at all worlds are dictated by the same semantical rules. Given the truth of $\square(P \land Q)$ at a world, for example, and the consequence that $P \land Q$ are true at all accessible worlds, we can safely say that $Q$ is true at those worlds as well, because the semantical rule for ‘$\land$’ applies uniformly at all worlds. This result transfers back to give us the truth of $\square Q$ at the original world. So if we want hypothetical modalities to behave uniformly, we should accept $K$ as a starting-point.

The original system of C.I. Lewis, $S3$, does not have the closure properties. The semantics for $S3$ is, therefore, not uniform. Worlds are segregated into the “normal” and “non-normal”. Normal worlds function as in the semantics for $K$, but at non-normal worlds, sentences of the form $\Diamond \alpha$ are always true, even if $\alpha$ is inconsistent. Thus, $S3$ violates Leibniz’s condition for possibility. Nonetheless, $S3$ does come close to closure:

If $\vDash S3 \alpha \prec \beta$, then $\vDash S3 \square \alpha \prec \square \beta$,

which is known as “Becker’s Rule.” Whether such a rule is sufficient for the purposes of formalizing notions of hypothetical possibility and necessity will not be discussed here. At any rate, one would want to use $S3$ to represent hypothetical modalities only if one had a reason to invoke conditions under which in some worlds that matter, anything is possible.

A result proved in the last section was that:

$\square \alpha \not\vDash K \Diamond \alpha$.

Should we say that what is hypothetically necessary is possible? We can see from the semantics of $K$ that the only reason the entailment fails is because there are worlds from the point of view of which nothing is possible. Now it may be thought that we need, for some reason, to consider such worlds in our reasoning about possibility. The idea would be that from the standpoint of that world, the condition embodied in the accessibility relation really cannot hold. What is odd about this non-consequence is that it seems to say that a sentence can hold at all worlds under a given hypothesis but not hold at some world under that hypothesis. But this is not what the semantics actually yields.

Rather, it can be the case that from the standpoint of a world, there is no way that a sentence can be true under a given hypothesis. But there is no way that it can be false under that hypothesis either, because the condition

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39 For Leibniz, this was God’s choice to create the world that he did.
40 This applies to the weaker systems $S1$ and $S2$ as well.
does not apply to anything accessible. This is what makes it “necessary”. And this simply reflects the nature of the meta-logic. At a world where nothing is hypothetically possible, the condition does not fail to hold with respect to something that is hypothetically possible. So at that world, such a thing has to be regarded as hypothetically necessary.

We saw in the last section that:

\[ \square \alpha \not\in K \alpha. \]

It may be thought that this consequence should hold in a logic of relative necessity. An argument for this can be made. What this consequence relation means is that whatever is necessary is also true. And surely whatever is necessary under some hypothesis is also true under that hypothesis.

We must be somewhat cautious here, though. What the consequence relation in question actually expresses is that whatever is necessary at a world is true at that same world. System \( K \) accommodates the fact that whatever is necessary at a world is true at those worlds that matter from the standpoint of that world, i.e., the accessible worlds. To put the matter in another way, we might wish to understand the hypothesis that governs the accessibility relation as sometimes not holding at a world at which the sentence is evaluated, but holding at all other worlds that meet the condition. For example, we might ponder what is necessary given worlds whose laws of nature are different from those that hold at the actual world.

### 7.2 Implicational Logic

The first informal application of \( MSL \) (indeed, the motivating one) was as a logic of the relation of logical implication. Thus for Lewis, \( \alpha \prec \beta \) was to signify “strict implication”, whose truth requires a logical connection between \( \alpha \) and \( \beta \), rather than the merely material connection between them in sentences of the form \( \alpha \supset \beta \). The semantical rule for this kind of sentence states that it is assigned \( T \) at a world \( w \) if and only if at all worlds accessible to \( w \), if \( \alpha \) has the value \( T \) there, then so does \( \beta \).

We saw in the last section that the accessibility relation best represents a notion of hypothetical necessity and possibility. Strict implication, on this reading, amounts to the requirement that there is no world at which a certain condition holds where \( \alpha \) is true and \( \beta \) is false. What strict implication in \( K \) represents, then, is a localized connection between the values of two sentences. This notion might be of some value. For example, it could reflect the necessity imposed by a law of nature at those worlds at which the law applies. It is still the case that if \( \alpha \) is true at an accessible world and \( \alpha \prec \beta \) is true at \( w \), then \( \beta \) is true at the accessible world \( w \).

The relation of strict implication cannot be said to hold in general, however, given the semantics for \( K \). One result is that \( \{ \alpha, \alpha \prec \beta \} \not\in K \beta \). A counter-example can be generated by giving \( \alpha \) and \( \alpha \prec \beta \) the value \( T \) at a world which is not accessible to itself. We can then give the value \( F \) to \( \beta \) at that world, since the value of \( \alpha \prec \beta \) is irrelevant to its value at that world. (It should
be noted that in all of Lewis’s systems, including his preferred system \( S2 \), this relation of entailment holds.) So \( K \) is not suitable for the representation of a general relation of logical connection. At best, we have the result that if \( \alpha \) and \( \alpha \prec \beta \) are \( K \)-valid, then \( \beta \) is \( K \)-valid. Closure of valid sentences over strict implication seems to be a necessary condition for any adequate representation of logical implication. The semantics for \( K \), then, at least accommodates this requirement.

It can be argued that aside from being in one respect too weak to represent logical implication, strict implication under the semantics for \( K \) is in another respect too strong. This is due to the presence of the two “paradoxes of strict implication”, of which Lewis was aware.

**Paradoxes of Strict Implication**

\[
\{\Box \alpha\} \vdash_K \beta \prec \alpha \\
\{\sim \Diamond \alpha\} \vdash_K \alpha \prec \beta
\]

On any interpretation and at any world at which \( \Box \alpha \) is given the value \( T \), \( \beta \prec \alpha \) must also be assigned \( T \). And on any interpretation and at any world at which \( \sim \Diamond \alpha \) is given the value \( T \), \( \alpha \prec \beta \) must also be assigned \( T \). A proof of the first “paradox” was given earlier. For the second, suppose that at an arbitrary world \( w \) in an arbitrary interpretation \( I \), \( v_I(\sim \Diamond \alpha,w) = T \). So \( v_I(\Diamond \alpha,w) = F \). Then at no world \( w \) accessible to \( w \), \( v_I(\alpha,w_i) = T \). Hence, at no such world, \( v_I(\alpha,w_i) = T \) and \( v_I(\beta,w_i) = F \). Therefore, at every accessible world, if \( v_I(\alpha,w_i) = T \), then \( v_I(\beta,w_i) = T \), in which case \( v_I(\alpha \prec \beta,w) = T \). Since the choice of \( w \) and \( I \) were arbitrary, the result holds at all worlds on all interpretations, and the entailment is thereby established.

This result is usually interpreted informally as meaning that a necessarily true sentence is strictly implied by any sentence, and a sentence that is not possibly true strictly implies any sentence. Critics have argued that no correct formalization of logical implication should permit the result. One line of argument is that in any such implication, the antecedent should be relevant to the consequent. But as can be seen from the form of the two strict implications, as well as from the semantical reasoning, the content of \( \beta \) is irrelevant to the truth of the strict implication sentence-form \( \alpha \prec \beta \) or \( \beta \prec \alpha \) which might be said to represent logical implication.

Lewis himself puts the point as saying that these consequence-relations do not reflect deductions which are either futile or gratuitous. Either we try to establish something on the basis of what we should not assert at all, or we try to establish something that is already established.

That to infer in such cases is affected with a sense of paradox, reflects the futility of drawing any inference when the premise is not only known false but is not even rational to suppose; and the gratuitous character of inferring what could be known to be true without reference to any premise. (*Symbolic Logic, Appendix III*, added January 5, 1959)

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This emphasizes the epistemic side of inference, that it is a tool for increasing our knowledge. Lewis suggests we might draw a possibility between “deducibility” and “inferability”, where deduction is purely logical and admits the paradoxes and inference is concerned with the “credibility” of premises and conclusion.

This distinction is related to Lewis’s intentions in his original article on modal logic, cited above. As has already been shown, \( \alpha \prec \beta \) is equivalent to \( \Box(\alpha \supset \beta) \). This is in turn equivalent to \( \Box(\sim \alpha \lor \beta) \), intensional disjunction. Lewis had applied both a logical and an epistemic criterion to extensional disjunction. From a purely logical standpoint, “its truth is independent of the truth of either member considered separately. Considered epistemically, (and “more accurately”), “its truth can be known, while it is still problematic which of its lemmas is the true one. It has a truth which is prior to the determination of the facts in question”. (“Implication and the Algebra of Logic”, pp. 524.)

In a case such as \( \Box(Q \lor (P \supset P)) \), the truth-value of the disjunction is known through that of the right disjunct ‘\( P \supset P \)’ alone. Since ‘\( Q \)’ is a sentence-letter, the value of the disjunction cannot be known from its logical form, which permits either the value \( T \) or the value \( F \). Ironically, Lewis’s own example of a sentence that fails the test for intensional disjunction, “Today is Monday or 2 + 2 = 4”, would be a true instance of \( \Box(\alpha \lor \beta) \), given that the right disjunct is taken to be a necessary truth. One cannot know it to be true if one had no idea of whether today is Monday or 2 + 2 = 4. So perhaps we should say that genuine intensional disjunction belongs to the domain of knowledge and credibility, while intensional disjunction as defined in MSL belongs to the domain of pure logical deduction.

Another equivalent to the sentence-form \( \alpha \prec \beta \) is \( \sim (\alpha \circ \sim \beta) \): the truth of \( \alpha \) is inconsistent with the falsehood of \( \beta \). If \( \beta \) must be false, then any sentence \( \alpha \) is inconsistent with its truth. Putting the matter this way is reminiscent of a common definition of a valid argument as one in which it is impossible that the premises be true and the conclusion false. This definition is negative, emphasizing the safety of a valid inference. In the view of Alan Ross Anderson and Nuel Belnap, two prominent critics of the informal interpretation of strict implication as logical implication, to treat implication this way is captures only a part of the meaning of the notion.\(^{41}\)

Another standard definition of validity is that the conclusion must be true given that the premises are true. This looks more positive than the first definition, but the two are treated as equivalent by most logicians. (Though they are not equivalent in the semantics for the system QPL of Modal Predicate Logic to be discussed below.) Those who do not think they are equivalent generally interpret the “given that” relation in some constructive way. That is, they think that the conclusion must be constructed from the premises in a certain way. So where \( \alpha \) is necessarily false, there may be no construction of a derivation of \( \beta \) from \( \alpha \). Consider the following derivation-schema.

\[^{41}\text{See Entailment: The Logic of Relevance and Necessity, Volume I, pp. 4-5.}\]
The assumption of $\sim \beta$ has nothing to do with what transpires in the next two steps, and the derivation of $\beta$, “given that” $\alpha$, does not really connect $\alpha$ and $\beta$ at all.

Lewis advanced a derivation that apparently makes such a connection in the specific case where ‘$Q$’ is derivable from the impossible ‘$P \& \sim P$’.

Which step are we to give up if we are to deny that ‘$Q$’ is logically implied by ‘$P \& \sim P$’? Each one seems to be a legitimate logical implication in itself. It is hard to see how one could give up & Elimination, so it appears that the choice comes down to Disjunctive Syllogism and $\lor$ Introduction. “relevance” (or “relevant”) logics give up the former and “connectivist” logics give up the latter.

Another possibility, in the spirit of Lewis’s first paper, is to disallow the use of Disjunctive Syllogism on the result of the use of $\lor$ Introduction. If we use $\lor$ Introduction, we already take it that one of the disjuncts is true. In the present case, we take ‘$P$’ to be true. So the disjunction is not intensional, by Lewis’s original criterion. It is not a matter of the logical relation between ‘$P$’ and ‘$Q$’ that the disjunction holds. Now given that we already take ‘$P$’ to be true, it could be argued, the only legitimate use of Disjunctive Syllogism would be to re-assert the truth of ‘$P$’, given the falsehood of ‘$Q$’ (which, of course, we do not have).

Consider Lewis’s example, “Either the today is Monday or $2 + 2 = 4$”. Suppose I assert this confidently solely on the grounds that “$2 + 2 = 4$” is a necessary truth. But then suppose further that I come upon the information that I have been deluded by the Cartesian Evil Demon into believing this, and in fact the sum of 2 and 2 is not 4. Given the reasoning of the preceding argument, I would be entitled to conclude, “Well, then, at least I know that today is Monday!” This is, of course, an absurd conclusion to draw. What I would do instead is to take back my assertion of the disjunction. So the idea is that in the presence of the negation of a sentence used to establish a disjunction by $\lor$ Introduction, the disjunction should not be used for further ends, but instead should be withdrawn.\footnote{Broadly speaking, such a system would not be monotonic. All the systems treated here}
more detail here, but it is worth noting that Lewis hints at this in the context of the inference/deducibility distinction that he suggested but did not endorse. “We might say that no inference is to be drawn from anything the assertion of which is rationally contraindicated” (Symbolic Logic, Appendix III, p. 514).

The three responses to Lewis’s derivation have in common the requirement that changes be made to the underlying non-modal Sentential Logic. This indicates the close relation between strict implication and the material conditional of which it is the necessitation. For better or worse, truth-functional Sentential Logic is the basis for the modal systems to be treated here. So we shall have to be content to assess the adequacy of strict implication without changing the underlying logic.

7.3 Deontic Logic

It does not seem controversial that people make inferences with premises involving obligation and permission. One might say, for example, that if it is obligatory both that I pay both federal income taxes and that I pay state income taxes, then it is obligatory that I pay federal income taxes. This appears to be a valid inference, whose underlying principle is closure of an obligation operator under logical consequence. So it looks initially as if system $K$ might be a good place to start in developing a logic of obligation, or deontic logic. Sentences whose main operator is the ‘□’ could then be considered analogous to sentences which indicate obligation for an agent. It is common practice to use a special piece of syntax, an “obligation” operator ‘O’ and a “permission” operator ‘P’ in deontic logic.

A natural interpretation of the ‘♦’ operator is as representing what is permissible. Given Duality, then, we can say that what is permitted (for an agent at a time in a situation) is what is not obligatorily not the case. If I am permitted to pay my federal income tax by check, I am under no obligation not to do so. Similarly, what is obligatory is not permissibly not the case. If I am obligated to pay federal income tax, I am not permitted not to pay it.

Given the initial plausibility of a deontic logic based on $K$, the next issue that arises is how to understand the accessibility relation in the formal semantics for $K$. On an analogy with hypothetical necessity, we can think of accessibility in the context of deontic logic as reflecting the conditions for right behavior that hold at a world—perhaps a moral law holding for all cognitive agents, or some civil law applying to all residents of a nation. On that view, worlds $w_i$ accessible to $w$ can be thought of as worlds which conform to the law that holds at $w$. For example, given that I live in a world where I am governed by the United States Tax Code, in every world compatible with that law, I pay federal income taxes. In some such worlds, I pay my taxes by check.

A nice way to look at deontic accessible worlds is in terms of ideals, as with Daniel Bonevac:

\begin{itemize}
  \item are monotonic.
\end{itemize}
We may imagine each possible world as looking to other possible worlds for its moral or practical values. These worlds need not be perfect, but they do need to be ideal in some respects. In particular, the worlds that our world holds up as ideal should be ones in which all the obligations that hold in our world are fulfilled. We can imagine ourselves as thinking of how things ought to be and measuring the current state of affairs, or our own behavior, by that yardstick. But not all worlds need have the same ideals. We can perhaps imagine that some worlds are better than ours, in that none of our obligations go unfulfilled there. Perhaps the moral standards of such a world, however, are so high that new obligations arise in that world. In contrast, there may be worlds so destitute that they hold up our world as an ideal. (*Deduction*, p. 313)

So it may be that what is not even permissible for me in the world in which I live would be obligatory from the standpoint of another world.

The condition on permissibility raises a problem for *K*, however. It seems intuitively right to say that what is obligatory is permissible, i.e., that \{Ωα\} entails Pα. If I must pay my federal income tax, then I am allowed to do so. But here we run up against the feature of *K* that this implication does not hold. If, from the standpoint of a given world, there are no ideal worlds, or worlds compatible with the federal Tax Code, then everything is obligatory, but nothing is permissible, at that world.

### 7.4 Doxastic Logic

The development of a logic of belief, or doxastic logic was initiated by Jaakko Hintikka, one of the originators of possible-worlds semantics. It may seem that the very idea of a doxastic logic is flawed. This is because on our ordinary conception, belief is not closed under logical consequence. Does it really follow from the fact that I believe something that I believe something else? When I was a child, tectonic plate theory had not yet been conceived, or at least had not yet been publicized. I believed (more than once) that I was eating an apple. Did I thereby believe that I was eating an apple or earthquakes are caused by the motion of tectonic plates?

This consideration is of importance in attempting to use the semantics for the ‘□’ operator in system *K* to represent belief by a subject at a time. Suppose we add an operator ‘B_s,t’ with an variable index ‘s’ for the subject and a variable index ‘t’ for the time. Given that \{P\} ⊨ K P ∨ Q, it follows that \{B_s,t P\} ⊨ K B_s,t (P ∨ Q). This case of closure yields the result questioned above. If we suppose that ‘B_s,t P’ is true at a world w, it follows that ‘B_s,t (P ∨ Q)’ is true at w. Hence, a subject will at any time believe all the consequences in *K* of what the subject believes.

The limiting case is where a sentence α is valid in *K*. In that case, it is valid in *K* that the subject believes α. The notion of its being valid that

43See his classic *Knowledge and Belief: The Logic of the Two Notions*. 

41
someone believes something, without any constraints on that subject’s even having considered it, is at best an idealization. So what K allows us to represent is a logic of the beliefs of a subject that is a kind of “logical saint.” Clearly, this is an idealized notion of belief. It might be thought of as a kind of deontic notion, a notion of what one should believe. The subject should believe what holds as a matter of the logic and what follows in the logic from what the subject believes.

If we view doxastic logic in this light, we can ask what to make of the accessibility relation. A sentence $B_{s,t}\alpha$ has the value T at a world w just in case $\alpha$ has the value T at all worlds accessible to w. We have been thinking of these worlds as embodying a condition that is assumed to hold. We will work with the condition that the accessible worlds are those worlds which for the subject are “live” possibilities, i.e., those which are not ruled out on the basis of the subject’s beliefs. Thus accessibility does nothing to explain what belief is, but instead is concerned with the consequences of what one believes.

For example, $\{B_{s,t}\alpha\} \vDash K \sim B_{s,t} \sim \alpha$. Given that the subject believes that the United States is the richest country in the world in 2001, it is true at all worlds compatible with what he believes that the United States is the richest country in the world in 2001. And it is false at all those worlds that the United States is not the richest country in the world in 2001. And so, the subject does not believe that the United States is not the richest world in 2001.

With this in mind, we may introduce a second operator to mirror the ‘♦’ operator. This operator is a kind of belief-compatibility operator, ‘P_{s,t}’. The semantics for this operator will be the same as that for the ‘♦’, so what is compatible with what someone believes is what holds at some world that is not ruled out by what one believes. So, for example, it is compatible with what I believe now that I will live to be 100 years old.

Given Duality, we have the result that $\{\sim P_{s,t} \sim \alpha\} \vDash K B_{s,t} \alpha$, i.e., that whatever is not incompatible with what one believes is believed. This result is initially quite implausible. But consider the extreme case, where an interpretation contains a dead end, a world to which no other world is accessible. The condition $\sim P_{s,t} \sim \alpha$ holds at any such world, and so $B_{s,t} \alpha$ also holds there. A dead-end signifies that there is nothing that is ruled out by one’s beliefs, and so everything is believed.

The half of Duality under consideration seems implausible because it seems that the truth of $\sim P_{s,t} \sim \alpha$ sets up a condition which requires a belief that $\alpha$ at that world. However, we must keep in mind that the accessible worlds are themselves defined by what the person believes. So the truth of $\sim P_{s,t} \sim \alpha$ at a dead-end is a reflection of the condition that the subject believes what he does, rather than being itself a condition that determines what the subject believes.

Now of course a logical saint is not the sort of subject that would believe everything, even what is inconsistent. So it seems desirable that the semantics for doxastic logic require that each world have at least one world accessible to it. In that way, whatever is false at that world is not believed by the subject. And in general, the subject will not be represented as believing anything inconsistent, since no world allows for any sentence and its negation to receive both of the
values \( T \) and \( F \).

### 7.5 Epistemic Logic

On the model of doxastic logic, Hintikka used an analogue of the \('\Box'\) operator, \('K_{s,t}'\), to indicate a subject’s knowing something at a time. We might wish to make inferences with knowledge-sentences as premises and conclusion, and hence to develop an *epistemic logic*. I might, then, infer from the premises, “I know now that this is a zebra” and “I know now that if this is a zebra, then this is not a mule,” to the conclusion, “I know now that this is not a mule.” Symbolically, we would have \( \{K_{i,n}Z, K_{i,n}(Z \supset \sim M)\}\)\(\vDash_K K_{i,n} \sim M\).

This is clearly an instance of closure of the knowledge operator over semantic entailment in \( K \). On most theories of knowledge, belief is a condition of knowledge, and so our remarks about closure in doxastic logic apply directly here. We will have to assume that our subject is a “logical saint.”

The accessibility relation can be handled similarly as well. Accessible worlds are those which are not ruled out by what one knows. Then the possibility operator \('F_{s,t}'\) indicates compatibility with what one knows. What was said about Duality in doxastic logic also applies here. If it is not the case, for all I know, that \( \sim \alpha \), then I know that \( \alpha \). We will want to move to a stronger condition to avoid the trivial truth of knowledge-sentences which is allowed by the presence of dead-end worlds in an interpretation.

Some philosophers have attacked the closure condition as leading to skepticism. For example, let \('H'\) symbolize ‘I am a victim of an elaborate hoax in which a mule is painted to look like a zebra’. We might then have the following instance of closure: \( \{K_{i,n}Z, K_{i,n}(Z \supset \sim H)\}\)\(\vDash K_{i,n} \sim H\). But, it is held, I do not know that I am not the victim of a hoax, because I cannot rule out the possibility of such deception given the evidence that I have: \( \sim K_{i,n} \sim H \). So from the standpoint of a world \( w \) (assuming that \( K_{i,n}(Z \supset \sim H) \) still has the value \( T \) there), there is an accessible world \( w_i \) at which \('H'\) is not assigned \( T \), and so at \( w, \sim K_{i,n}Z \) is assigned \( T \) (assuming as well that \( Z \supset \sim H \) is also true at \( w \)).

One way to counter this objection and save closure is to limit the scope of accessible worlds. In the standard informal interpretation of the accessibility relation, the accessible worlds are all situations compatible with what I know. We could think of the accessible worlds instead as situations which rule out a limited range of possibilities relative to what one knows. We can call these worlds *relevant alternatives*. Some things incompatible with what I know “don’t matter” when evaluating what I do know.

Applying this to the present case, worlds at which \('H'\) is assigned \( T \) might not be made relevant, and so they are not deemed accessible. So the fact that \('H'\) is assigned \( T \) at those worlds does not affect the value of \( K_{i,n} \sim H \), and hence does not affect the value of \( K_{i,n}Z \) at \( w \). This result does not prove

\[\text{This famous example is taken from Fred Dretske's paper, “Epistemic Operators”}. \] \[\text{It will be developed in what follows.} \]

\[\text{See, for example, the articles reprinted in Part Three of Skepticism: A Contemporary Reader, by Keith DeRose and Ted A. Warfield.} \]
anything in the theory of knowledge. It only allows us to represent an aspect
of a “relevant alternatives” theory formally. As was stated in connection with
doxastic logic, whatever condition is represented by the accessibility relation
is taken from a prior understanding of belief or knowledge. Whether knowing
requires ruling out all alternatives or only all relevant alternatives, or whether
I know that I am not the victim of a hoax in the case under consideration, are
matters of debate among epistemologists.

7.6 Temporal Logic

The modern logic of time, *temporal logic*, was initially studied extensively by
Arthur Prior.\(^46\) Time is quite naturally adapted to modal interpretation. The
ancient Megarian philosopher Diodorus Cronus interpreted necessity and possi-
bility themselves temporally, according to Cicero.

Diodorus defines the possible as that which either is or will be, the
impossible as that which, being false, will not be true, the necessary
as that which, being true, will not be false, and the non-necessary
as that which either is already or will be false. (Cited in Kneale and

We shall here work with an informal interpretation of the modalities of MSL
that is somewhat simpler than Diodorus’s. It is limited to representing the
future, and so can be enriched considerably by adding modalities representing
the past and the present.

An analogue of the possibility operator in temporal logic is ‘F’, which is
understood as indicating that something will be the case in the future. We
have seen that closure of possibility over entailment holds for single possibility
sentences in \(K\). So we have it in temporal logic that if \(\{\alpha\}\models_K \beta\) then \(\{\alpha\}\models_K F\beta\):
if \(\beta\) is entailed by \(\{\alpha\}\), then if \(\alpha\) will be the case, then \(\beta\) will be the case. This
seems to be reasonable with respect to the temporal notion we want to represent.
So the notion of a logic of time, unlike that of belief and knowledge, is plausible
from the outset.

The possible worlds are obviously going to be understood as moments of
time, arranged in a linear sequence. Worlds accessible to a given world should be
those which stand later in the sequence. Here, the flexibility of the accessibility
relation is an asset. We would not in general want to take into account worlds
representing moments in the past in considering whether \(\alpha\) will be the case from
the present standpoint, for example.

We can add an operator ‘G’ analogous to the necessity operator. A sentence
of the form \(G\alpha\) is true at a moment if and only if \(\alpha\) is true at all future moments.
Given Duality, \(G\alpha\) is equivalent to \(\sim F\sim \alpha\). Since our semantics admits of two
and only two values, we can say that \(\sim \alpha\) will not be the case if and only if \(\alpha\)
will always be the case.

\(^{46}\)See *Time and Modality and Past, Present and Future*. 
Here we run immediately into the limitations of $K$. We clearly want time to be sequential, which requires some restrictions on accessibility. Certainly we want to hold that $G\alpha$ implies $F\alpha$, that what always will be the case will be the case. So it is clear from the outset that a stronger system will be needed to represent these temporal modalities. In the next chapter, we will develop systems that seem appropriate for the temporal and other modalities we have been examining in this section.