Derived Rules for Quantifiers

Contradictions and Isolated Names

- The use of ∃ E often leaves us in a situation in which we derive a pair of contradictory sentences containing a name which is isolated in the current subderivation.

1	$\vert\; (\forall x) \sim Fx$	P
2	$\vert\;\vert\; (\exists x)Fx$	A
3	$\vert\;\vert\; {}^{a}\vert\; Fa$	A
4	$\vert\;\vert\;\vert\; \sim Fa$	1 ∀ E

- We would like to be able to "bring out" the contradiction, to apply ∼ I on step 2 to get ∼(∃x)Fx.

Two Derived Rules

- To shorten the derivation that we will produce on the next slide, we introduce two variants of existing rules.
  - Negation Introduction

$\vert\;\vert\; \mathbf{X}$	Assumption
$\vert\;\vert\; \ldots$	
$\vert\;\vert\; \mathbf{Y} \& \sim \mathbf{Y}$	
$\vert\; \sim \mathbf{X}$	$\sim$ I

  - Reductio

$\vert\;\vert\; \sim \mathbf{X}$	Assumption
$\vert\;\vert\; \ldots$	
$\vert\;\vert\; \mathbf{Y} \& \sim \mathbf{Y}$	
$\vert\; \mathbf{X}$	RD

Bringing Out the Contradiction

1	$(\forall x) \sim Fx$	P
2	$(\exists x)Fx$	A
3	a Fa	A
4	$\sim Fa$	1 $\forall$ E
5	$\sim (P\& \sim P)$	A
6	Fa	3 R
7	$\sim Fa$	4 R
8	$P\& \sim P$	5-7 RD
9	$P\& \sim P$	2 3-8 $\exists$ E
10	$\sim (\exists x)Fx$	2-9 $\sim$ I

Another Derived Rule

- A derived rule **Contradiction** can shorten the work further.

$\mathbf{X}$	Already Derived
$\sim \mathbf{X}$	Already Derived
$\mathbf{Y}$	CD

- It is best to pick a standard contradictory sentence, such as 'A & ∼A' for use in the problem cases such as the one we have just described.
- This leaves us with the shortened derivation in the following frame.

The Shortest Version of the Derivation

1	$(\forall x) \sim Fx$	P
2	$(\exists x)Fx$	A
3	a Fa	A
4	$\sim Fa$	1 $\forall$ E
5	$P\& \sim P$	3 4 CD
6	$P\& \sim P$	2 3-5 $\exists$ E
7	$\sim (\exists x)Fx$	2-6 RD

Duality

- Derivations can be shortened significantly through the use of four rules which reflect the **duality** of the quantifiers.
- The existential quantifier '$(\exists x)$' could be eliminated in favor of '$\sim(\forall x)\sim$', and the universal quantifier '$(\forall x)$' could be eliminated in favor of '$\sim(\exists x)\sim$'.
- The syntax of some systems of Predicate Logic contains only one quantifier symbol, usually the universal.
- It is easy to see the semantic basis for duality.
  - Some variant satisfies an open sentence if and only if it is not the case that all variants do not satisfy that open sentence.
  - All variants satisfy an open sentence if and only if it is not the case that there is a variant that does not satisfy that open sentence.

Negated Quantifier Sequences

- The logical equivalences involving negated quantifies we proved earlier in the term are just variants of duality involving double negation.
  - $\sim(\forall x)$, $\sim\sim(\exists x)\sim$, $(\exists x)\sim$.
  - $\sim(\exists x)$, $\sim\sim(\forall x)\sim$, $(\forall x)\sim$.
  - $(\forall x)\sim$, $\sim(\exists x)\sim\sim$, $\sim(\exists x)$.
  - $(\exists x)\sim$, $\sim(\forall x)\sim\sim$, $\sim(\forall x)$.
- The logical equivalence of sentences which begin with these strings of operators allows us to introduce four new derived rules.

Negated Quantifier Rules

$\sim(\forall \mathbf{u})(\ldots \mathbf{u} \ldots)$	Already Derived
$(\exists \mathbf{u}) \sim (\ldots \mathbf{u} \ldots)$	$\sim \forall$

$\sim(\exists \mathbf{u})(\ldots \mathbf{u} \ldots)$	Already Derived
$(\forall \mathbf{u}) \sim (\ldots \mathbf{u} \ldots)$	$\sim \exists$

$(\exists \mathbf{u}) \sim (\ldots \mathbf{u} \ldots)$	Already Derived
$\sim(\forall \mathbf{u})(\ldots \mathbf{u} \ldots)$	$\exists \sim$

$(\forall \mathbf{u}) \sim (\ldots \mathbf{u} \ldots)$	Already Derived
$\sim(\exists \mathbf{u})(\ldots \mathbf{u} \ldots)$	$\forall \sim$

Using Negated Quantifier Rules

- The Negated Quantifier rules, like the derived rules for Sentence Logic in this text, apply to whole sentences only and not to their internal parts.
- They could be used as **rules of replacement** within sentences, but this makes proving their soundness more difficult.
- Although the use of these rules makes many derivations easier, it is frequently the case that derivations that do not use them are more elegant and yield more insight into why the conclusion follows from the premises.