Definite Descriptions

Upper and Lower Bounds

- We can set upper and lower bounds on the number of objects in the domain which satisfy an open sentence.
  - At least two, at least three, at least four, . . .
  - At most one, at most two, at most three, . . .
- These bounds can be specified by the use of multiple quantifiers and identity symbols.

Lower Bounds

- To say there are at least *n* objects satisfying an open sentence, one needs to use *n* existential quantifiers and state (if necessary) non-identities for all the distinct variables they contain.
  - At least one P: $(\exists x)Px$
  - At least two Ps: $(\exists x)(\exists y)((Px \mathbin{\&} Py) \mathbin{\&} x \neq y)$.
  - At least three Ps: $(\exists x)(\exists y)(\exists z)[(Px \mathbin{\&} Py \mathbin{\&} Pz) \mathbin{\&} (x \neq y \mathbin{\&} y \neq z \mathbin{\&} x \neq z)]$.

Upper Bounds

- To say there are at most *n* objects satisfying an open sentence, one needs to use *n+1* universal quantifiers and state identity for at least one pair of distinct variables the quantifiers contain.
  - At most one P: $(\forall x)(\forall y)((Px \mathbin{\&} Py) \supset x = y)$.
  - At most two Ps: $(\forall x)(\forall y)(\forall z)[(Px \mathbin{\&} Py \mathbin{\&} Pz) \supset (x = y \vee y = z \vee x = z)]$.
  - At most three Ps: $(\forall x)(\forall y)(\forall z)(\forall w)[(Px \mathbin{\&} Py \mathbin{\&} Pz \mathbin{\&} Pw) \supset (x = y \vee x = z \vee x = w \vee y = z \vee y = w \vee z = w)]$.

Exact Quantities

- To say that there are exactly *n* objects satisfying an open sentence, one must state that there are at least *n* and at most *n* such objects.
- It is possible to combine these specifications in a single sentence.
  - Exactly one P: $(\exists x)(Px \mathbin{\&} (\forall y)(Py \supset x = y))$.
  - Exactly two Ps: $(\exists x)(\exists y)[((Px \mathbin{\&} Py) \mathbin{\&} x \neq y) \mathbin{\&} (\forall z)(Pz \supset (x = z \vee y = z))]$.

E Shriek

- It is traditional for logicians to use a special quantifier symbol, '$\exists!$', pronounced "E shriek", to indicate that exactly one member of the domain satisfies an open sentence.
- Note that the text places the '!' after the variable in a quantifier, which is contrary to common usage and therefore not adopted here.
- There is exactly one P: $(\exists x)(Px \,\&\, (\forall y)(Py \supset x = y))$. $(\exists! x)Px$.
- The exclamation point can indicate shrieking, from which the name may have been taken.
- For any open sentence $\mathbf{P}(\mathbf{u})$ with $\mathbf{u}$ a free variable, $(\exists!\mathbf{u})\mathbf{P}(\mathbf{u})$ is shorthand for $(\exists\mathbf{u})[\mathbf{P}(\mathbf{u}) \,\&\, (\forall\mathbf{v})(\mathbf{P}(\mathbf{v}) \supset \mathbf{v} = \mathbf{u})]$, where $\mathbf{v}$ is free for $\mathbf{u}$ in $\mathbf{P}(\mathbf{u})$ [i.e., $\mathbf{v}$ is free at all the places where $\mathbf{u}$ is free in $\mathbf{P}(\mathbf{u})$].

Definite Descriptions

- Grammatically, a **definite description** is a term which can serve as a subject of a predicate or as a term in a relation.
  - The first person to cross the finish line wins the race.
  - The tallest member of the Houston Rockets is taller than the tallest member of the Sacramento Kings.
- Semantically, a definite description is supposed to pick out exactly one object, the one and only one that falls under the description it contains.
- In the second sentence, on February 14, 2005, the first definite description refers to Yao Ming and the second to Greg Ostertag.
- If all the occurrences of definite descriptions refer successfully, then the sentence is true just in case the relevant objects satisfy the open sentence containing the description.

Non-Denoting Definite Descriptions

- In 1905, Bertrand Russell raised the question of the truth-value of sentences containing non-denoting definite descriptions ("On Denoting").
  - The present king of France is bald.
- There is no present king of France, so the definite description does not denote.
- Russell decided to treat any such sentence as false, on the grounds that the use of the definite description indicates the existence of what it ostensibly denotes.
- Thus, the sentence 'The present king of France is bald' is false.
- An alternative approach is to declare that the sentence has no truth-value when a definite description it contains fails to denote.

Symbolizing Definite Descriptions

- Russell held that definite descriptions should be symbolized **contextually**.
- That is, there is no special symbol for the description itself, but any sentence containing a definite description can be symbolized by other means.
- One way to do this is to say that there is exactly one thing satisfying the condition of the description, and that any such thing satisfies the predicate of the sentence itself.
  - Fx: x is present king of France, Bx: x is bald.
  - $(\exists!x)Fx \ \& \ (\forall x)(Fx \supset Bx)$.
  - $(\exists x)(Fx \ \& \ (\forall y)(Fy \supset x = y)) \ \& \ (\forall x)(Fx \supset Bx)$.
- Russell himself symbolized it in the following (equivalent) way:
  - $(\exists x)(Fx \ \& \ (\forall y)(Fy \supset x = y) \ \& \ Bx)$

Syntactical Proof of Implication of Russell's Transcription by Teller's

1		$(\exists x)(Fx\&(\forall y)(Fy \supset x = y))\&(\forall x)(Fx \supset Bx)$	P
2		$(\forall x)(Fx \supset Bx)$	1 & E
3		$(\exists x)(Fx\&(\forall y)(Fy \supset x = y))$	1 & E
4	a	$Fa\&(\forall y)(Fy \supset a = y))$	A
5		$Fa \supset Ba$	2 ∀ E
6		Fa	4 & E
7		Ba	5 6 ⊃ E
8		$Fa\&(\forall y)(Fy \supset a = y)\&Ba$	4 7 & I
9		$(\exists x)(Fx\&(\forall y)(Fy \supset x = y)\&Bx)$	8 ∃ I
10		$(\exists x)(Fx\&(\forall y)(Fy \supset x = y)\&Bx)$	1 4-9 ∃ E

An Instructive Error

- The example (1a) for symbolizing definite descriptions in *A Modern Formal Language Primer* Volume II, p. 153, is incorrect.
- The first symbolization in the last slide is a correction of this error.
- The erroneous formulation was due to a faulty reading of E shriek.

- $(\exists!x)(Fx \,\&\, Bx)$ should be rewritten as:
- $(\exists x)[(Fx \,\&\, Bx) \,\&\, (\forall y)((Fy \,\&\, By) \supset x = y)]$, and not:
- $(\exists x)[(Fx \,\&\, Bx) \,\&\, (\forall y)(Fy \supset x = y)]$

- The mistaken formulation would be equivalent to the Russell formulation, but for the occurrence of 'Bx' in the antecedent of the conditional.
- The lesson is to be sure to use "rules for rewriting" carefully.

A Counterexample

- The formulation is too weak.
- There might be exactly one object in the domain that satisfies both the description and the predicate of the sentence, but more than one object that satisfies the description itself.
- Consider a domain with two objects, the number 1 and the number 3.
- The description 'the odd number' would fail to denote given such a domain, since more than one object satisfies the description 'odd number'.
- However, 'The odd number is less than two' is true on the formulation, where Ox: x is odd, Tx: x is less than two.
- $(\exists x)[(Ox \,\&\, Tx) \,\&\, (\forall y)((Oy \,\&\, Ty) \supset x = y)]$

Another Formalization of Definite Descriptions

- Since 'the . . .' functions as a term in English, we might wish to represent it as a term in Predicate Logic.
- The term would be complex (as is a function term).
- The descriptive part of the description is written as an open sentence $\mathbf{P}(\mathbf{u})$.
- This is prefixed by (The $\mathbf{u}$), to form the term (The $\mathbf{u}$)$\mathbf{P}(\mathbf{u})$.
  - 'The odd number' is symbolized as '(The x)Ox'. 'The odd number is less than two' would then be 'T(The x)Ox'.
- Generally, where $\mathbf{P}(\mathbf{u})$ and $\mathbf{Q}(\mathbf{u})$ are open sentences with $\mathbf{u}$ as the only free variable, $\mathbf{Q}$[(The $\mathbf{u}$)$\mathbf{Pu}$] is shorthand for: $(\exists!\mathbf{u})[\mathbf{P}(\mathbf{u}) \,\&\, (\forall\mathbf{u})(\mathbf{P}(\mathbf{u}) \supset \mathbf{Q}(\mathbf{u})]$.

Wide and Narrow Scope

- Consider the sentence ‘The present king of France is not bald’.
- This sentence is ambiguous.
- It may mean that it is false to say that the present king of France is bald, in which case the negation has **wide scope**, while description has **narrow scope**.
  - $\sim(\exists!x)(Fx \,\&\, (\forall x)(Fx \supset Bx))$
- Or it may mean that the present king of France lacks the property being bald, in which case the negation has **narrow scope** while the description has **wide scope**.
  - $(\exists!x)(Fx \,\&\, (\forall x)(Fx \supset \sim Bx))$

Dealing with Ambiguity

- The narrow scope reading of the description is true if there is no present king of France, since ‘$(\exists!x)(Fx \,\&\, (\forall x)(Fx \supset Bx))$’ is false.
- The wide scope reading of the description is false if there is no king of France, since no object satisfies the condition specified by ‘$(Fx \,\&\, (\forall x)(Fx \supset \sim Bx))$’.
- The abbreviation of the first reading involves a convention that reflects the wide scope of the negation symbol:
  - $\sim[B(\text{The } x)Fx]$
- The abbreviation of the second reading reflects the narrow scope of the negation symbol, which governs ‘Bx’ in the original sentence’.
  - $\sim B(\text{The } x)Fx$
- The text refers to the “negated predicate” ‘$\sim B$’, but in fact it is the open sentence ‘Bx’ that is negated.