Solutions to Selected Exercises Using Formal Semantics

2-6 d). Show that the following argument is valid:

$$\frac{(\forall x)(Bx \ \& \ Lxe)}{(\forall x)Bx}$$

Proof. Let **I** be an arbitrary interpretation in which '(∀x)(Bx & Lxe)' is true. Let **d** be an arbitrary variable assignment based on **I**. Suppose that **d** satisfies '(∀x)(Bx & Lxe)'. Then for every item **o** in the domain of **I**, **d**[**o**/x] satisfies 'Bx & Lxe'. Therefore, **d**[**o**/x] satisfies both 'Bx' and 'Lxe'. Since **d**[**o**/x] satisfies 'Bx' for all members **o** of the domain, **d** satisfies '(∀x)Bx'. Since the choice of **d** is arbitrary, '(∀x)Bx' is satisfied by all variable assignments based on **I**, and so '(∀x)Bx' is true in **I**. Since the choice of **I** is arbitrary, '(∀x)Bx' is true in all interpretations in which '(∀x)(Bx & Lxe)' is true, so that the argument is valid.

6-3 b). Show that the following sentence is a logical truth:

$$(\forall x)(Gx \lor \sim Gx)$$

Proof. Let **I** be an arbitrary interpretation, **D** be the domain of **I**, **v** be the valuation function in **I**, and **d** be an arbitrary variable assignment based on **I**. For any object **o** in **D**, either $\mathbf{o} \in \mathbf{v}(G)$ or $\mathbf{o} \notin \mathbf{v}(G)$. If $\mathbf{o} \in \mathbf{v}(G)$, then **d**[**o**/x] satisfies 'Gx', and hence it satisfies 'Gx ∨ ∼Gx'. If $\mathbf{o} \notin \mathbf{v}(G)$, then **d**[**o**/x] does not satisfy 'Gx'. In that case, **d**[**o**/x] satisfies '∼Gx', and hence it satisfies 'Gx ∨ ∼Gx'. Therefore, since every member **o** in **D** either is or is not in **v**(G), **d**[**o**/x] satisfies 'Gx ∨ ∼Gx' for all **o** in **D**. Then **d** satisfies '(∀x)(Gx ∨ ∼Gx)'. Since the choice of **d** and **v** are arbitrary, the sentence is true in **I**. Since the choice of **I** is arbitrary, the sentence is true in all interpretations, i.e., is a logical truth.

Alternative Proof (Indirect). Suppose that '(∀x)(Gx ∨ ∼Gx)' is not a logical truth. Then there is some interpretation **I** in which it is false. Therefore, there is some variable assignment **d** based on **I** such that **d** does not satisfy '(∀x)(Gx ∨ ∼Gx)'. Thus, there is a member **o** of the domain of **I** such that $\mathbf{d}[\mathbf{o}/x]$ does not satisfy 'Gx ∨ ∼Gx'. Then $\mathbf{d}[\mathbf{o}/x]$ does not satisfy 'Gx' and it does not satisfy '∼Gx', in which case **o** does satisfy 'Gx'. So $\mathbf{d}[\mathbf{o}/x]$ does and does not satisfy 'Gx', which is a contradiction. We conclude that our assumption was false, and that '(∀x)(Gx ∨ ∼Gx)' is in fact a logical truth.

6-4 d). Show that the following sentence is a contradiction.

(∀x)(∃y)(Mxy & ∼Mxy)

Proof. Suppose that there is an interpretation **I** in which '(∀x)(∃y)(Mxy & ∼Mxy)' is true. Then all variable assignments **d** satisfy '(∀x)(∃y)(Mxy & ∼Mxy)'. It follows that for all members **o** of the domain of **I**, $\mathbf{d}[\mathbf{o}/x]$ satisfies '(∃y)(Mxy & ∼Mxy)'. Therefore, for some object $\mathbf{o}_1$ in the domain of **I**, $\mathbf{d}[\mathbf{o}/x, \mathbf{o}_1/y]$ satisfies 'Mxy & ∼Mxy'. So $\mathbf{d}[\mathbf{o}/x, \mathbf{o}_1/y]$ satisfies 'Mxy' and it satisfies '∼Mxy'. But in that case $\mathbf{d}[\mathbf{o}/x, \mathbf{o}_1/y]$ does not satisfy 'Mxy', and so it both satisfies and does not satisfy 'Mxy', a contradiction. So the assumption that the sentence is true in some interpretation is false, and '(∀x)(∃y)(Mxy & ∼Mxy)' is true on no interpretation, which means that it is a contradiction.

6-5 d). Show that the following set of sentences is inconsistent:

(∃x)Dx, (∀x)(Dx ⊃ (∀y)(∀z)Ryz), and (∃x)(∃y)∼Rxy

Proof. Suppose there is an interpretation **I** in which each of the following sentences is true: '(∃x)Dx', '(∀x)(Dx ⊃ (∀y)(∀z)Ryz)', and '(∃x)(∃y)∼Rxy'. Then each of the sentences is satisfied by all variable assigments **d** based on **I**. Since **d** satisfies '(∃x)Dx', there is at least one object $\mathbf{o}_1$ in the domain of **I**, such that $\mathbf{d}[\mathbf{o}_1/x]$ satisfies 'Dx'. Since **d** satisfies '(∃x)(∃y)∼Rxy', there is at least one object $\mathbf{o}_2$ in the domain such that $\mathbf{d}[\mathbf{o}_2/x]$ satisfies '(∃y)∼Rxy', and it follows that there is at least one object $\mathbf{o}_3$ in the domain such that $\mathbf{d}[\mathbf{o}_2/x, \mathbf{o}_3/y]$ satisifes '∼Rxy', in which case $\mathbf{d}[\mathbf{o}_2/x, \mathbf{o}_3/y]$ does not satisfy 'Rxy'. Then $\langle \mathbf{o}_2, \mathbf{o}_3 \rangle \notin \mathbf{I}(R)$.

Now given the fact that **d** satisfies '(∀x)(Dx ⊃ (∀y)(∀z)Ryz)', it follows that

$\mathbf{d}[\mathbf{o}_1/\text{x}]$ satisfies 'Dx ⊃ (∀y)(∀z)Ryz'. From this and the fact that $\mathbf{d}[\mathbf{o}_1/\text{x}]$ satisfies 'Dx', it follows that $\mathbf{d}[\mathbf{o}_1/\text{x}]$ satisfies '(∀y)(∀z)Ryz'. Therefore, $\mathbf{d}[\mathbf{o}_1/\text{x}, \mathbf{o}_2/\text{y}]$ satisfies '(∀z)Ryz'. And it follows once more that $\mathbf{d}[\mathbf{o}_1/\text{x}, \mathbf{o}_2/\text{y}, \mathbf{o}_3/\text{z}]$ satisfies 'Ryz'. Hence $\langle \mathbf{o}_2, \mathbf{o}_3 \rangle \in \mathbf{I}(\text{R})$.

This yields a contradiction, and so there is no interpretation in which all the sentences are true, which means that the set of sentences is inconsistent.

6-6 d). Show that the folowing pair of sentences is not logically equivalent.

(∃x)(Px & Qx), (∃x)Px & (∃x)Qx

Proof. We will generate an interpretation in which '(∃x)Px & (∃x)Qx' is true and '(∃x)(Px & Qx)' is false. Let I be an interpretation whose domain is $\{1, 2\}$, and let $\langle 1 \rangle \in \text{v(P)}$ but $\langle 1 \rangle \notin \text{v(Q)}$. Further, let $\langle 2 \rangle \notin \text{v(P)}$ and $\langle 2 \rangle \in \text{v(Q)}$. (You can think of 'P' as expressing the property of being odd, and 'Q' the property of being even, though this fact does not enter into the proof.) In that case, for an arbitrary variable assignment **d**, $\mathbf{d}[1/\text{x}]$ satisfies 'Px', and so **d** satisfies '(∃x)Px'. Similarly, $\mathbf{d}[2/\text{x}]$ satisfies 'Qx', so **d** satisfies '(∃x)Qx'. Then the conjunction '(∃x)Px & (∃x)Qx' is satisfied by **d**. Since **d** is arbitrary, the sentence is true in I.

On the other hand, neither 1 nor 2 belongs to both v(P) and v(Q). $\mathbf{d}[1/\text{x}]$ does not satisfy 'Qx' and $\mathbf{d}[2/\text{x}]$ does not satisfy 'Px'. So there is no x-variant of **d** which satisfies both 'Px' and 'Qx'. It follows that there is no x-variant of **d** which satisfies 'Px & Qx'. (Alternatively, it could be said that there is no object **o** in the domain such that $\mathbf{d}[\mathbf{o}/\text{x}]$ satisfies both 'Px' and 'Qx', in which case $\mathbf{d}[\mathbf{o}/\text{x}]$ does not satisfy 'Px & Qx' for any **o** at all.) Therefore, **d** does not satisfy '(∃x)(Px & Qx)'. It follows that not all variable assignments satisfy the sentence, and so it is false in I.

We have proved that in our interpretation one of the sentences of the pair is true, and the other is false, so the two sentences are not logically equivalent.