Meta-Logic of Predicate Logic

Derivability and Valdity

- An argument of Predicate Logic ***Z***, therefore **X** (or ***Z*****X**) is **valid** if and only if in every interpretation in which all the members of ***Z*** are true, **X** is also true.
  - {(∀x)(Fx ⊃ Gx), Fa} \\Ga
- **X** is **derivable** from premises ***Z*** if and only if there is a proof of **X** from only premises ***Z*** using the rules of inference.

1	(∀x)(Fx ⊃ Gx)	P
2	Fa	P
3	Fa ⊃ Ga	1 ∀ E
4	Ga	2 3 ⊃ E

Soundness and Completeness

- ***Z*** ⊢ **X** if and only if **X** is derivable from **Z**.
  - '⊢' is a metalogical symbol known as the 'turnstyle'.
  - It indicates a purely syntactical relation between the members of ***Z*** and **X**.
- ***Z*** ⊨ **X** if and only if the argument ***Z*** **X** is valid.
  - '⊨' is a metalogical symbol known as the 'double turnstyle'.
  - It indicates a semantical relation between the members of ***Z*** and **X**.
- A system of formal proof is **sound** if and only if for all ***Z*** and all **X**, if ***Z*** ⊢ **X**, then ***Z*** ⊨ **X**.
- A system of formal proof is **complete** if and only if for all ***Z*** and all **X**, if ***Z*** ⊨ **X**, then ***Z*** ⊢ **X** (proved by Kurt Gödel in 1930).

Undecidability

- Given that a system of formal proof is complete, there is a proof for every valid argument.
- In Predicate Logic, there is no purely syntactical way to determine whether a proof for any given argument exists or semantical way to determine whether any given argument is valid.
- There is no mechanical **decision procedure** for determining derivability or validity in a finite number of steps.

- Thus Predicate Logic is **undecidable** (proved by Alonzo Church in 1936).
- However, various fragments of Predicate Logic are decidable.
  - For example, the set of sentences of Predicate Logic in which no predicate has more than one argument.

Second-Order Predicate Logic

- Predicate Logic, originallly proposed by Frege, can be extended by allowing quantification over predicates.
  - For example, we might formulate an axiom of identity.
  - $(\forall x)(\forall y)(x = y \equiv (\forall P)(Px \equiv Py))$.
- Semantically, the quantifiers would range over properties or sets, rather than the objects in the domain over which first-order quantifiers range.
- Second-order logic cannot be given a proof system which is sound and complete relative to its interpretation, though fragments of it can be given a sound and complete proof system.
- As with first-order logic, the fragment of second-order logic consisting of none but one-place predicates is decidable.
- It is possible to construct logics of higher, even infinite, order.