Transcription and Restricted Quantifiers

Differences Among Quantifier Expressions

- Some English quantifier expressions are neutral with respect to the range of objects whose quantity they express.
  - Any, every, all, whatever - some
  - Anything, everything - something
  - There is, at least one is
- Other quantifier expressions apply only to a limited range of objects.
  - Anyone, anybody - someone, somebody (persons)
  - Anywhere, everywhere - somewhere (places)
  - Whenever, always when - sometimes (times)

Limiting the Domain

- One way of transcribing sentences with restricted quantifiers is to limit the domain to the objects to which the quantifiers are supposed to apply.
  - D = x: x is a person, so '$(\forall x)$' and '$(\exists x)$' apply only to persons.
  - 'Everybody is happy' is transcribed as '$(\forall x)Hx$', where Hx: x is happy.
- However, this only allows us to talk about items in the domain, so that in the last example, we could not transcribe 'Everybody is happy sometimes'.

Restricted Quantifiers

- One way to transcribe sentences with a mixture of types of quantifier expressions is to create a new kind of quantifier in Predicate Logic: the **restricted quantifier**.
- We choose a predicate letter to symbolize the restricted range of objects.
  - 'P' stands for the set of all persons.
  - We write '$(\forall x)_P$' for 'everyone', and '$(\exists x)_P$' for 'someone'.
  - 'Everyone is happy' is transcribed as '$(\forall x)_P Hx$'.

  We can mix restricted quantifiers to symbolize sentences containing more than one limited quantifier expression.
  - 'T' stands for the set of all times.
  - 'Everyone is happy sometimes' is transcribed as '$(\forall x)_P(\exists y)_T Hxy$', where Hxy: x is happy at y.

Semantics for Restricted Quantifiers

- Tarski-style semantics can be used to specify satisfaction-conditions for sentences with restricted quantifiers.
- For each restriction represented by a one-place predicate **S**, we generate the set **r**(**S**) from **v**(**S**) by stripping off the angle brackets.
  - Let v(P) = {⟨Adam⟩, ⟨Eve⟩}; r(P) = {Adam, Eve}.
- Then we say that **d** satisfies $(\exists\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ if and only if for some object $\mathbf{o} \in \mathbf{r}(\mathbf{S})$, **d**[**o**/**u**] satisfies **P**(**u**).
  - Let v(B) = {⟨Adam⟩}; then d[Adam/x] satisfies 'Bx', so d satisfies '$(\exists x)_P Bx$'.
- Similarly, **d** satisfies $(\forall\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ if and only if for all objects $\mathbf{o} \in \mathbf{r}(\mathbf{S})$, **d**[**o**/**u**] satisfies **P**(**u**).

Eliminating Restricted Quantifiers

- Restricted quantifiers can be eliminated in favor of other constructions without change in truth-value.
- $(\exists\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ is equivalent to $(\exists\mathbf{u})(\mathbf{S}(\mathbf{u})\ \&\ \mathbf{P}(\mathbf{u}))$.
  - '$(\exists x)_P Bx$' is equivalent to '$(\exists x)(Px\ \&\ Bx)$'.
- $(\forall\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ is equivalent to $(\forall\mathbf{u})(\mathbf{S}(\mathbf{u}) \supset \mathbf{P}(\mathbf{u}))$.
  - '$(\forall x)_P Bx$' is equivalent to '$(\forall x)(Px \supset Bx)$'.
- The replacement of one form for the other (when authorized) may occur in an internal part of a sentence.
  - '$(\exists x)_{\mathbf{P}} Lxe \supset (\exists x)_{\mathbf{P}} Lex$' is equivalent to '$(\exists x)(Px\ \&\ Lxe) \supset (\exists x)(Px\ \&\ Lex)$'.

Proof of Equivalence for Restricted Existentials

- $(\exists\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ is true in **I** if and only if (iff) it is satisfied by all variable assigments **d** based on **I**.
- Let **I** be an arbitrary interpretation and **d** an arbitrary variable assigment based on **I**.
- **d** satisfies $(\exists\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ iff some object $\mathbf{o} \in \mathbf{r}(\mathbf{S})$, **d**[**o**/**u**] satisfies **P**(**u**),
- iff for some object **o** in **D**, **d**[**o**/**u**] satisfies **P**(**u**), and the one-tuple $\langle\mathbf{o}\rangle \in \mathbf{v}(\mathbf{S})$ [by the definition of **r**(**S**)],
- iff for some object $\mathbf{o} \in \mathbf{D}$, **d**[**o**/**u**] satisfies **P**(**u**) and **d**[**o**/**u**] satisfies **S**(**u**),

- iff for some object $\mathbf{o}$ in $\mathbf{D}$, $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisifies $\mathbf{S}(\mathbf{u})$ & $\mathbf{P}(\mathbf{u})$,
- iff $\mathbf{d}$ satisfies $(\exists\mathbf{u})(\mathbf{S}(\mathbf{u})\ \&\ \mathbf{P}(\mathbf{u}))$,
- Since $\mathbf{d}$ is arbitrary, $(\exists\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ is true in $\mathbf{I}$ iff $(\exists\mathbf{u})(\mathbf{S}(\mathbf{u})\ \&\ \mathbf{P}(\mathbf{u}))$ is true in $\mathbf{I}$, QED.

Proof of Equivalence for Restricted Universals

- To save space, only the core of the proof is presented; the other steps are trivial.
- $\mathbf{d}$ satisfies $(\forall\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ iff for all objects $\mathbf{o} \in \mathbf{r}(\mathbf{S})$, $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisfies $\mathbf{P}(\mathbf{u})$,
- iff for all objects $\mathbf{o} \in \mathbf{D}$, if $\mathbf{o} \in \mathbf{r}(\mathbf{S})$, then $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisifies $\mathbf{P}(\mathbf{u})$,
- iff for all objects $\mathbf{o} \in \mathbf{D}$, if the one-tuple $\langle\mathbf{o}\rangle \in \mathbf{v}(\mathbf{S})$, then $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisfies $\mathbf{P}(\mathbf{u})$,
- iff for all objects $\mathbf{o} \in \mathbf{D}$, if $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisfies $\mathbf{S}(\mathbf{u})$, then $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisfies $\mathbf{P}(\mathbf{u})$,
- iff for all objects $\mathbf{o} \in \mathbf{D}$, $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisifes $\mathbf{S}(\mathbf{u}) \supset \mathbf{P}(\mathbf{u})$,
- iff, $\mathbf{d}$ satisfies $(\forall\mathbf{x})(\mathbf{S}(\mathbf{u}) \supset \mathbf{P}(\mathbf{u}))$, QED.

Negated Restricted Quantifiers

- The following two logical equivalences hold, with a proof of the first below.
  - $\sim(\forall\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ and $(\exists\mathbf{u})_{\mathbf{S}} \sim\mathbf{P}(\mathbf{u})$
  - $\sim(\exists\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ and $(\forall\mathbf{u})_{\mathbf{S}} \sim\mathbf{P}(\mathbf{u})$
- $\mathbf{d}$ satisfies $\sim(\forall\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$ iff $\mathbf{d}$ does not satisfy $(\forall\mathbf{u})_{\mathbf{S}}\mathbf{P}(\mathbf{u})$,
- iff it is not the case that for all $\mathbf{o} \in \mathbf{D}$, if $\mathbf{o} \in \mathbf{r}(\mathbf{S})$, then $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisfies $\mathbf{P}(\mathbf{u})$,
- iff for some $\mathbf{o} \in \mathbf{D}$, it is not the case that if $\mathbf{o} \in \mathbf{r}(\mathbf{S})$, then $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisfies $\mathbf{P}(\mathbf{u})$,
- iff for some $\mathbf{o} \in \mathbf{D}$, $\mathbf{o} \in \mathbf{r}(\mathbf{S})$ and $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ does not satisfy $\mathbf{P}(\mathbf{u})$,
- iff for some $\mathbf{o} \in \mathbf{D}$, $\mathbf{o} \in \mathbf{r}(\mathbf{S})$, and $\mathbf{d}[\mathbf{o}/\mathbf{u}]$ satisfies $\sim\mathbf{P}(\mathbf{u})$,
- iff $\mathbf{d}$ satisfies $(\exists\mathbf{u})_{\mathbf{S}} \sim\mathbf{P}(\mathbf{u})$, QED.