Classical Correspondence Theory for Basic Modal Logic

Aldo Antonelli
University of California, Davis
http://aldo-antonelli.org

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Part I

Systems of Modal Logic

1. Introduction

Modal Logic deals with *modal propositions* and the entailment relations among them. Examples of modal propositions are the following:

- ▻ It is necessary that $2+2=4$.
- ▻ It is necessarily possible that it will rain tomorrow.
- ▻ If it is necessarily possible that φ then it is possible that φ.

Possibility and necessity are not the only modalities: other unary connectives are also classified as modalities, for instance, "it ought to be the case that φ," "It will be the case that φ," "*A* knows that φ," or "*A* believes that φ."

Modal logic makes its first appearance in Aristotle's *De Interpretatione*: he was the first to notice that necessity implies possibility, but not viceversa; that possibility and necessity are inter-definable; that If $\varphi \wedge \psi$ is possibly true then φ is possibly true and ψ is possibly true, but not conversely; and that if $\varphi \to \psi$ is necessary, then if φ is necessary, so is ψ.

The first modern approach to modal logic was the work of C.I. Lewis, culminating with Lewis & Langford, *Symbolic Logic* (1932). Lewis & Langford were unhappy with the representation of implication by means of the material conditional: $\varphi \to \psi$ is a poor substitute for "φ implies ψ." Instead, they proposed to characterize implication as "Necessarily: if φ then ψ," symbolized as $\varphi \prec \psi$. In trying to sort out the different properties, Lewis indetified five different modal systems, $\mathsf{S1}, \dots, \mathsf{S4}, \mathsf{S5}$, the last two of which are still in use.

The approach of Lewis & Langford was purely *syntactical*: they identified reasonable axioms and rules and investigated wat was provable with those means. A semantic approach remained elusive for a long time, until a first attempt was made by R. Carnap in *Meaning and Necessity* (1947) using the notion of a *state description*, i.e., a collection of atomic sentences (those that are "true" in that state description). After lifting the truth definition to arbitrary sentences φ, Carnap defines φ to be *necessarily true* if it is true in all state descriptions. Carnap's approach could not handle *iterated* modalities, in that sentences of the form "Possibly necessarily ... possibly φ" always reduce to the innermost modality.

The major breakthrough in modal semantics came with S. Kripke's article "A Completeness Theorem in Modal Logic" (JSL 1959). Kripke based his work on Leibniz's idea that a statement is necessarily true if it is true "at all possible worlds." This idea, though, suffers from the same drawbacks as Carnap's, in that the truth of statement at a world w (or a state description s) does not depend on w at all. So Kripke assumed that worlds are related by an *accessibility relation* R, and that a statement of the form "Necessarily φ" is true at a world w if and only if φ is true at all worlds w' *accessible from* w. Semantics that provide some version of this approach are called Kripke semantics and made possible the tumultuous development of modal logics (in the plural).

When interpreted by the Kripke semantics, modal logic shows us what *relational structures* look like "from the inside." A relational structure is just a set equipped with a binary relation (for instance, the set of students in the class ordered by their social security number is a relational structure). But in fact relational structures come in all sorts of domains: besides relative possibility of states of the world, we can have epistemic states of some agent related by epistemic possibility, or states of a dynamical system with their state transitions, etc. Modal logic can be used to model all of these: the first give us ordinary, alethic, modal logic; the others give us epistemic logic, dynamic logic, etc.

In these notes we focus on one particular angle, known to modal logicians as "correspondence theory." One of the most significant early discoveries of Kripke's is that many properties of the accessibility relation R (whether it is transitive, symmetric, etc.) can be characterized *in the modal language* itself by means of appropriate "modal schemas." Modal logicians say, for instance, that the reflexivity of R "corresponds" to the schema "If necessarily φ, then φ". These notes explore mainly the correspondence theory of a number of classical systems of modal logic (e.g., $\mathsf{S4}$ and $\mathsf{S5}$) obtained by a combination of the schemas D, T, B, $\mathsf{4}$, and $\mathsf{5}$. The material in these notes is derived from various sources, first and foremost Chellas' still outstanding *Modal Logic* (1980). To the best of the author's knowledge, nothing in here is original, except the typesetting.

2. Syntactic Preliminaries

1. Definition: The language $\mathscr{L}$ of modal logic comprises the following:

- ▻ Finitely or denumerably many propositional variables $p, q, r, \dots$ possibly with subscripts.
- ▻ A propositional constant, $\bot$.
- ▻ Truth functional connectives: $\neg$ and $\to$.
- ▻ A modal connective for necessity: $\Box$.
- ▻ Parentheses: (and).

We also define $\mathscr{L}_0$ to be the non-modal fragment of $\mathscr{L}$, i.e., the language obtained by dropping $\Box$.

2. Definition: A string of symbols from $\mathscr{L}$ is a *formula* if and only if it is a propositional variable or constant, or if it is of the form $(\neg\varphi)$ or $(\varphi \to \psi)$ or $(\Box\varphi)$, where φ and ψ are formulas. We denote the set of formulas over $\mathscr{L}$ by $\mathsf{Fml}_{\mathscr{L}}$.

3. Definition: We introduce conventions and abbreviations as usual:

- ▻ We drop parentheses around $(\neg\varphi)$ and $(\Box\varphi)$, with the understanding that $\neg$ and $\Box$ apply to the shortest formula to their right.
- ▻ We drop outermost pairs of parentheses in formulas.
- ▻ $\varphi \lor \psi$ abbreviates $\neg\varphi \to \psi$.
- ▻ $\varphi \,\&\, \psi$ abbreviates $\neg(\varphi \to \neg\psi)$.
- ▻ $\Diamond\varphi$ abbreviates $\neg\Box\neg\varphi$.

4. Definition: Where φ is a modal formula in $\mathsf{Fml}_{\mathscr{L}}$, all of whose propositional variables are among

$p_1, \ldots, p_n$, and $\theta_1, \ldots, \theta_n$ are in $\mathsf{Fml}_{\mathscr{L}}$, we define $\varphi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}]$ as the result of simultaneously substituting each θ_i for p_i in φ. Formally, this is a definition by recursion on φ:

▷ If φ is atomic, say $\varphi = q$, then $\varphi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}] = \theta_i$ if q is p_i for some $i \leq n$, and $\varphi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}] = q$ otherwise (including the case $\varphi = \bot$)).

▷ If $\varphi = \neg\psi$, then $\varphi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}] = \neg(\psi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}])$.

▷ If $\varphi = \psi \to \chi$, then $\varphi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}] = (\psi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}]) \to (\chi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}])$.

▷ If $\varphi = \Box\psi$, then $\varphi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}] = \Box(\psi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}])$.

The formula $\varphi[{}^{\theta_1}/_{p_1}, \ldots, {}^{\theta_n}/_{p_n}]$ is called a *substitution instance* of φ.

Below we will be interested in substitution instances of *non-modal* formulas, i.e., formulas in $\mathsf{Fml}_{\mathscr{L}_0}$. Definition 4 covers this case as well, of course: the recursion clase for $\Box\psi$ is just never applied. Now recall the following from propositional logic:

5. Definition: An assignment is a function v from the propositional variables into the set of two truth values $\{0, 1\}$ (representing "false" and "true," respectively). Each assignment can be lifted to a function $\overline{v} : \mathsf{Fml}_{\mathscr{L}_0} \to \{0, 1\}$, by recursion as follows:

$$
\begin{aligned}
\overline{v}(p) &= v(p);\\
\overline{v}(\neg\psi) &= 1 - \overline{v}(\psi);\\
\overline{v}(\varphi \to \psi) &= 1 - (\overline{v}(\varphi) \times (1 - \overline{v}(\psi))).
\end{aligned}
$$

(Obviously, $x = 1$ if and only if $1 - x \neq 1$.)

6. Definition: A modal-free formula $\varphi \in \mathsf{Fml}_{\mathscr{L}_0}$ is a *tautology* if and only if $\overline{v}(\varphi) = 1$ for every assignment v.

3. Kripke Semantics

7. Definition: A *model* for a language $\mathscr{L}$ is a triple $M = (W, R, V)$, where:

▷ W is a non-empty set of "worlds."

▷ R is a binary accessibility relation on W.

▷ V is a function assigning to each propositional variable p a set $V(p)$ of possible worlds.

8. Definition: We define the notion φ is true at w in M, written $M \models_w \varphi$, where φ is a formula, M a

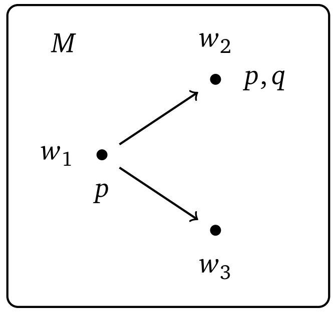


Figure 1: A simple model.

model and w a world in M, by recursion on φ:

$$
\begin{array}{rcl}
M \models_w p & \Leftrightarrow & w \in V(p); \\
M \models_w \bot & & \text{never;} \\
M \models_w \neg\varphi & \Leftrightarrow & M \not\models_w \varphi \\
M \models_w \varphi \to \psi & \Leftrightarrow & \text{either } M \not\models_w \varphi \text{ or } M \models_w \psi; \\
M \models_w \Box\varphi & \Leftrightarrow & \forall w' [Rww' \Rightarrow M \models_{w'} \varphi].
\end{array}
$$

The great advantage of Kripke semantics is that models can be represented by means of simple diagrams, such as the one in Figure 1. Worlds are represented by nodes, and world w' is accessible from w precisely when there is an arrow from w to w'. Moreover, we write p next to a world precisely when $w \in V(p)$.

9. Exercise: Consider the model of Figure 1. Which of the following hold?

(a) $M \models_{w_1} q$;	(d) $M \models_{w_1} \Box(p \vee q)$;	(g) $M \models_{w_1} \Diamond q$;
(b) $M \models_{w_3} \neg q$;	(e) $M \models_{w_3} \Box q$;	(h) $M \models_{w_1} \Box q$;
(c) $M \models_{w_1} p \vee q$;	(f) $M \models_{w_3} \Box\bot$;	(i) $M \models_{w_1} \neg\Box\Box\neg q$.

10. Definition: If Γ is a set of formulas and φ a formula, then Γ *entails* φ — in symbols: $\Gamma \models \varphi$ — if and only if for every model $M = (W, R, V)$ and world $w \in W$, if $M \models_w \psi$ for every $\psi \in \Gamma$, then $M \models_w \varphi$. If Γ contains a single formula ψ, then we write $\psi \models \varphi$.

11. Exercise: Show that $\Box(p \to q) \not\models p \to \Box q$ and $p \to \Box q \not\models \Box(p \to q)$.

12. Exercise: Let $M = (W, R, V)$. Show that $M \models_w \Diamond\varphi$ if and only if $\exists w' [Rww'$ and $M \models_{w'} \varphi]$.

13. Definition: A formula φ is *true in* a model $M = (W, R, V)$ — written $M \models \varphi$ — if and only if $M \models_w \varphi$ for every $w \in W$.

14. Proposition: Facts about truth in a model:

(a) If $M \models \varphi$ then $M \not\models \neg\varphi$, but *not* vice-versa.

(b) If $M \models \varphi \to \psi$ then $M \models \varphi$ only if $M \models \psi$, but *not* vice-versa.

Proof. For (a): If $M \models \varphi$ then φ is true at all worlds in W, and since $W \neq \varnothing$, it can't be that $M \models \neg\varphi$, or else φ would have to be both true and false at some world. Conversely, if $M \not\models \neg\varphi$ then φ is true at some world $w \in W$; it does not follow that $M \models \varphi$. For instance, in the model of Figure 1, $M \not\models \neg p$, but it does not follow that $M \models p$.

For (b): assume $M \models \varphi \to \psi$ and $M \models \varphi$; to show $M \models \psi$ let $w \in W$ be an arbitrary world. Then $M \models_w \varphi \to \psi$ and $M \models_w \psi$, so $M \models_w \varphi$, and since w was arbitrary, $M \models \varphi$. The converse fails: we need to find a model M such that $M \models \varphi$ only if $M \models \psi$, but $M \not\models \varphi \to \psi$. Consider again the model of Figure 1: $M \not\models p$ and hence (vacuously) $M \models p$ only if $M \models q$. However, $M \not\models p \to q$, as p is true but q false at w_1. ■

15. Definition: A formula φ is *valid* in a class $\mathscr{C}$ of models if it is true in every model in $\mathscr{C}$ (i.e., true at every world in every model in $\mathscr{C}$). If φ is valid in $\mathscr{C}$ we write $\mathscr{C} \models \varphi$, and we write $\models \varphi$ if φ is valid in the class of *all* models.

16. Proposition: If φ is valid in $\mathscr{C}$ it is also valid in each class $\mathscr{C}' \subseteq \mathscr{C}$.

17. Exercise: Show that the following are valid:

(a) $\models \Box p \to \Box(q \to p)$;

(b) $\models \Box\neg\bot$;

(c) $\models \Box p \to (\Box q \to \Box p)$.

18. Definition: A modal formula $\psi \in \mathsf{Fml}_{\mathscr{L}}$ is a *tautological instance* if and only if there is a tautology $\varphi \in \mathsf{Fml}_{\mathscr{L}_0}$ and formulas $\theta_1, \ldots \theta_n \in \mathsf{Fml}_{\mathscr{L}}$ such that $\psi = \varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$.

19. Theorem: Suppose $\varphi \in \mathsf{Fml}_{\mathscr{L}_0}$ be a modal-free formula all of whose propositional variables are among $p_1, \ldots, p_n$, and let $\theta_1, \ldots \theta_n \in \mathsf{Fml}_{\mathscr{L}}$. Then for any assignment v, any model $M = (W, R, V)$, and any $w \in W$ such that $v(p_i) = 1$ if and only if $M \models_w \theta_i$: $\overline{v}(\varphi) = 1$ if and only if $M \models_w \varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$.

Proof. By induction on φ. If φ is atomic, then by the hypothesis it must be some p_i, whence:

$$\overline{v}(p_i) = 1 \Leftrightarrow v(p_i) = 1 \Leftrightarrow M \models_w \theta_i \Leftrightarrow M \models_w \varphi[\theta_1/p_1, \ldots, \theta_n/p_n].$$

Assume the theorem holds for ψ and χ, in order to show that it holds for $\neg\psi$ and $\psi \to \chi$ (we do not consider the case for $\Box\psi$ since φ is modal-free). For the first of these, we have:

$$\begin{aligned}
\overline{v}(\neg\psi) = 1 &\Leftrightarrow \overline{v}(\psi) \neq 1 && \text{def. } \overline{v}; \\
&\Leftrightarrow M \not\models_w \psi && \text{ind. hyp.}; \\
&\Leftrightarrow M \models_w \neg\psi.
\end{aligned}$$

The case for $\varphi \to \psi$ is similar, using two inductive hypotheses:

$$\begin{aligned}
\overline{v}(\psi \to \chi) = 1 &\Leftrightarrow 1 - (\overline{v}(\psi) \times (1 - \overline{v}(\chi))) = 1, && \text{def. } \overline{v};\\
&\Leftrightarrow \overline{v}(\psi) \neq 1 \text{ or } \overline{v}(\chi) = 1;\\
&\Leftrightarrow M \not\models_w \psi \text{ or } M \models_w (\chi), && \text{ind. hyp.} \times 2;\\
&\Leftrightarrow M \models_w \psi \to \chi, && \text{def. } \models.
\end{aligned}$$ ■

20. Corollary: All tautological instances are valid.

Proof. Contrapositively, suppose φ is such that $M \not\models_w \varphi[{}^{\theta_1}/_{p_1}, \dots, {}^{\theta_n}/_{p_n}]$, for some model M and world w. Define an assignment v such that $v(p_i) = 1$ if and only if $M \models_w \theta_i$ (and v assigns arbitrary values to $q \notin \{p_1, \dots, p_n\}$). Then by the theorem $\overline{v}(\varphi) \neq 1$, so φ is not a tautology. ■

21. Proposition: If φ is valid, then so is $\Box\varphi$.

Proof. Assume $\models \varphi$. To show $\models \Box\varphi$ let $M = (W, R, V)$ be a model and $w \in W$. If Rww' then $M \models_{w'} \varphi$, since φ is valid, and so also $M \models_w \Box\varphi$. Since M and w were arbitrary, $\models \Box\varphi$. ■

22. Exercise: Let $M = (W, R, V)$ be a model, and suppose $u, v \in W$ are such that:

- ▷ $u \in V(p) \Leftrightarrow v \in V(p)$; and
- ▷ for all $w \in W$: Ruw if and only if Rvw.

Using induction on formulas, show that for all formulas φ: $M \models_u \varphi$ if and only if $M \models_v \varphi$.

4. Schemas

23. Definition: A *schema* is a set S of formulas comprising all and only the substitution instances of some formula χ in $\mathsf{Fml}_{\mathscr{L}}$:

$$\mathsf{S} = \{\psi : \exists\theta_1, \dots, \exists\theta_n \, (\psi = \chi[{}^{\theta_1}/_{p_1}, \dots, {}^{\theta_n}/_{p_n}])\}.$$

The formula χ is called the *characteristic* formula of the schema, and it is unique up to a renaming of the propositional variables. A formula φ is an *instance* of a schema if it is a member of the set S.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting $\varphi, \psi, \dots$ for the atomic components of χ. So, for instance, the following denote schemas: φ, $\varphi \to \Box\varphi$, $\varphi \to (\psi \to \varphi)$, etc. Note that $\varphi = \mathsf{Fml}_{\mathscr{L}}$.

24. Definition: A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

25. Theorem: The following schema K is valid: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

Proof. We need to show that all instances of the schema are true at every world in every model. So let $M = (W, R, V)$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume

Valid Schemas	*Invalid Schemas*
$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	$\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$
$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$	$(\Diamond\varphi \,\&\, \Diamond\psi) \to \Diamond(\varphi \,\&\, \psi)$
$\Box(\varphi \,\&\, \psi) \leftrightarrow (\Box\varphi \,\&\, \Box\psi)$	$\varphi \to \Box\varphi$
$\Box\varphi \to \Box(\psi \to \varphi)$	$\Box\Diamond\varphi \to \psi$
$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$	$\Box\Box\varphi \to \Box\varphi$
$\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$	$\Box\Diamond\varphi \to \Diamond\Box\varphi.$

Figure 2: Valid and (or?) invalid schemas.

the antecedent is true to show that consequent is true as well. In this case, let $M \models_w \Box(\varphi \to \psi)$ and $M \models_w \Box\varphi$. We need to show $M \models \Box\psi$. So let w' be arbitrary such that Rww'. Then by the first assumption $M \models_{w'} \varphi \to \psi$ and by the second assumption $M \models_{w'} \varphi$. It follows that $M \models_{w'} \psi$. Since w' was arbitrary, $M \models_w \Box\psi$. ■

26. Proposition: Show that if φ and $\varphi \to \psi$ are true at a world in a model then so is ψ. Hence, valid formulas are closed under *Modus Ponens*.

27. Exercise: Show that none of the following schemas are valid:

D: $\Box\varphi \to \Diamond\varphi$;

T: $\Box\varphi \to \varphi$;

B: $\varphi \to \Box\Diamond\varphi$;

4: $\Box\varphi \to \Box\Box\varphi$;

5: $\Diamond\varphi \to \Box\Diamond\varphi$.

28. Exercise: Prove that the schemas in the first column of Figure 2 are valid and those in the second column are not valid.

29. Exercise: Decide whether the following schemas valid or invalid:

(a) $(\Diamond\varphi \to \Box\psi) \to (\Box\varphi \to \Box\psi)$;

(b) $\Diamond(\varphi \to \psi) \vee \Box(\psi \to \varphi)$.

5. Frame correspondence

30. Definition: If $\mathscr{C}$ is a class of models, we write $\mathscr{C} \models \varphi$, "$\varphi$ is valid in $\mathscr{C}$," to mean that φ is true in every model $M \in \mathscr{C}$.

31. Exercise: Show that $\varphi \to \Box\varphi$ is valid in the class $\mathscr{C}$ of models $M = (W, R, V)$ where $W = \{w\}$. Similarly, show that $\psi \to \Box\varphi$ and $\Diamond\varphi \to \psi$ are valid in the class of models $M = (W, R, V)$ where $R = \varnothing$.

32. Definition: A *frame* is a pair $F = (W, R)$ where W is a non-empty set of worlds and R a binary relation on W. A model M is *based on* a frame $F = (W, R)$ if and only if $M = (W, R, V)$.

33. Definition: If $\mathscr{F}$ is a class of frames, we write $\mathscr{F} \models \varphi$, "$\varphi$ is valid in $\mathscr{F}$," to mean that φ is true in every model M based on a frame $F \in \mathscr{F}$. Similarly for $\mathscr{C} \models \varphi$ where $\mathscr{C}$ is a class of models.

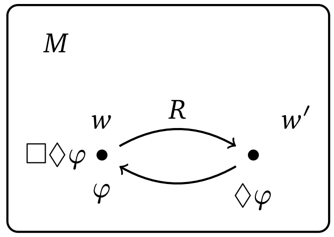


Figure 3: The argument from symmetry.

The reason frames are interesting is that correspondence between schemas and properties of the accessibility relation R is at the level of frames, *not of models*.

34. Remark: Obviously, if a formula or a schema is valid, i.e., valid with respect to the class of *all* models, it is also valid with respect to any class $\mathscr{C}$ of models or class $\mathscr{F}$ of frames.

35. Definition: We single out the five following potential properties of an accessibility relation:

R is called ...	*... if it satisfies:*
"serial"	$\forall u \exists v\, Ruv$;
"reflexive"	$\forall w\, Rww$;
"symmetric"	$\forall u \forall v (Ruv \Rightarrow Rvu)$;
"transitive"	$\forall u \forall v \forall w (Ruv \,\&\, Rvw \Rightarrow Ruw)$;
"euclidean"	$\forall w \forall u \forall v (Rwu \,\&\, Rwv \Rightarrow Ruv)$.

36. Theorem: Let $M = (W, R, V)$ be a model. Then:

- ▷ If R is serial then schema D, i.e., $\Box\varphi \to \Diamond\varphi$, is true in M;
- ▷ If R is reflexive then schemaT, i.e., $\Box\varphi \to \varphi$, is true in M;
- ▷ If R is symmetric then schema B, i.e., $\varphi \to \Box\Diamond\varphi$, is true in M;
- ▷ If R is transitive then schema 4, i.e., $\Box\varphi \to \Box\Box\varphi$, is true in M;
- ▷ If R is euclidean then schema 5, i.e., $\Diamond\varphi \to \Box\Diamond\varphi$, is true in M.

Proof. Here is the case for B: to show that the schema is true in a model we need to show that all of its instances are true all worlds in the model. So let $\varphi \to \Box\Diamond\varphi$ be a given instance of B, and let $w \in W$ be an arbitrary world. Suppose the antecedent φ is true at w, in order to show that $\Box\Diamond\varphi$ is true at w. So we need to show that $\Diamond\varphi$ is true at all w' accessible from w. Now, for any w' such that Rww' we have, using the hypothesis of symmetry, that also $Rw'w$ (see Figure 3). Since $M \models_w \varphi$, we have $M \models_{w'} \Diamond\varphi$. Since w' was an arbitrary world such that Rww', we have $M \models_w \Box\Diamond\varphi$. ■

Notice that the converse implications of Theorem 36 do not hold: it's not true that if a model verifies a schema, then the accessibility relation of that model has the corresponding property (a counterexample is provided by Exercise 37).

If R is . . .	*. . . the following schema is true in M:*
partially functional: $\forall w \forall u \forall v(Rwu \,\&\, Rwv \Rightarrow u = v)$	$\Diamond\varphi \to \Box\varphi$;
functional: $\forall u \exists! v Ruv$	$\Diamond\varphi \leftrightarrow \Box\varphi$;
weakly dense: $\forall u \forall v(Ruv \Rightarrow \exists w(Ruw \,\&\, Rwv))$	$\Box\Box\varphi \to \Box\varphi$;
weakly connected: $\forall w \forall u \forall v((Rwu \,\&\, Rwv) \Rightarrow (Ruv \vee u = v \vee Rvu))$	L: $\Box((\varphi \,\&\, \Box\varphi) \to \psi) \vee \Box((\psi \,\&\, \Box\psi) \to \varphi)$;
weakly directed: $\forall w \forall u \forall v((Rwu \,\&\, Rwv) \Rightarrow \exists t(Rut \,\&\, Rvt)$	G: $\Diamond\Box\varphi \to \Box\Diamond\varphi$;

Figure 4: Five more correspondence facts.

37. Exercise: Let $M = (W, R, V)$ be a model such that $W = \{u, v\}$, where worlds u and v are related by R: i.e., both Ruv and Rvu. Suppose that for all p: $u \in V(p) \Leftrightarrow v \in V(p)$. Show that:

(a) For all φ: $M \models_u \varphi$ if and only if $M \models_v \varphi$ (use induction on φ).

(b) Schema T is true in M.

Conclude that, since M is not reflexive (it is, in fact, *irreflexive*), the converse of Theorem 36 fails in the case of T (similar arguments can be given for some — though not all — the other schemas mentioned in Theorem 36).

Even though the converse implications of Theorem 36 fail, they hold if we replace "model" by "frame": for the properties considered in Theorem 36, it *is* true that if a schema is valid in a *frame* then the accessibility relation of that frame has the corresponding property. In fact, even more is true in the case of D.

38. Exercise: Show that any model where schema D is true is serial (Hint: take $\varphi = \neg\bot$).

Although we will focus on the five classical schemas D, T, B, 4, and 5, we record in the following exercise a few more correspondences.

39. Exercise: Let $M = (W, R, V)$ be a model. Show that if R satisfies the left-hand properties of Figure 4, the corresponding right-hand schemas are true in M.

We now proceed to establish the full correspondence results for frames. We wil consider T, B, 4 and 5, as the case for D already follows from Exercise 38.

40. Theorem: Recall that a schema S is valid in a frame if each of its instances is true in every model based on that frame. Then:

(a) If T is valid in a frame F, then F is reflexive.

(b) If B is valid in a frame F, then F is symmetric.

(c) If 4 is valid in a frame F, then F is transitive.

(d) If 5 is valid in a frame F, then F is euclidean.

Proof. The strategy is to devise, for each frame F, a valuation that will ensure that the frame has the desired property (provided the corresponding schema is true).

For (a): suppose T is valid in $F = (W,R)$, let $w \in W$ be an arbitrary world; we need to show Rww. Fix a propositional variable p and let $u \in V(p)$ if and only if Rwu (when q is other than p, $V(q)$ is arbitrary, say $V(q) = \varnothing$). Let $M = (W,R,V)$. By construction, for all u such that Rwu: $M \models_u p$, and hence $M \models_w \Box p$. But by hypothesis $\Box p \to p$, an instance of T, is true at w, so that $M \models_w p$, but by definition of V this is possible only if Rww.

For (b): suppose B is valid in $F = (W,R)$, and let $u, v \in W$ be arbitrary worlds such that Ruv; we need to show that Rvu. Fix a propositional variable p, define V such that $w \in V(p)$ if and only if Rvw (and V is arbitrary otherwise). Let $M = (W,R,V)$. Notice that the following instance of B: $\neg p \to \Box\Diamond\neg p$, is equivalent to $\Diamond\Box p \to p$. Now, by definition of V, $M \models_w p$ for all w such that Rvw, and hence $M \models_v \Box p$. Since Ruv, also $M \models_u \Diamond\Box p$, and since B is valid in F, also $M \models_u \Diamond\Box p \to p$. It follows that $M \models_u p$, whence Rvu, as required.

For (c): suppose $\mathsf{4}$ is valid in $F = (W,R)$, and let $u, v, w \in W$ be arbitrary worlds such that Ruv and Rvw; we need to show that Ruw. Fix a propositional variable p, define V such that $z \in V(p)$ if and only if Ruz (and V is arbitrary otherwise). Let $M = (W,R,V)$. By definition of V, $M \models_z p$ for all z such that Ruz, and hence $M \models_u \Box p$. But by hypothesis $\Box p \to \Box\Box p$, an instance of $\mathsf{4}$, is true at u, so that $M \models_u \Box\Box p$. Since Ruv and Rvw, we have $M \models_w p$, but by definition of V this is possible only if Ruw, as desired.

For (d) we proceed contrapositively, assuming that the frame $F = (W,R)$ is not euclidean, and falsifying an instance of $\mathsf{5}$. Suppose there are worlds u, v, w such that Rwu and Rwv but not Ruv. Fix a propositional variable p and define V such that for all worlds z, $z \in V(p)$ if and only if it is *not* the case that Ruz. Let $M = (W,R,V)$. Then by hypothesis $M \models_v p$ and since Rwv also $M \models_w \Diamond p$. However, there is no world y such that Ruy and $M \models_y p$ so $M \models_u \neg\Diamond p$. Since Rwu, it follows that $M \not\models_w \Box\Diamond p$, so that the instance of $\mathsf{5}$, $\Diamond p \to \Box\Diamond p$ fails at w. ■

Theorem 40 also shows that the properties can be combined: for instance if both B and $\mathsf{4}$ are valid in F then the frame is both symmetric and transitive, etc. This is useful because the classical systems $\mathsf{S4}$ and $\mathsf{S5}$ are, in fact, just the systems characterized as $\mathsf{KT4}$ and $\mathsf{KTB4}$.

We now record some properties of accessibility relations (in fact, these notions apply to arbitrary binary relations).

41. Proposition: Let R be a binary relation on a set W; then:

(i) If R is reflexive, then it is serial.

(ii) If R is symmetric, then it is transitive if and only if it is euclidean.

(iii) If R is symmetric or euclidean then it is weakly directed (it has the "diamond property").

(iv) If R is euclidean then it is weakly connected.

(v) If R is functional then it is serial.

42. Definition: A binary relation R on W is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation R on W is *universal* if and only if Ruv for all $u, v \in W$.

43. Exercise: Show that the following are equivalent:

(i) R is an equivalence relation;

(ii) R is reflexive and euclidean;

(iii) R is serial, symmetric, and transitive;

(iv) R is serial, symmetric, and euclidean.

Exercise 43 is the semantic counterpart to Exercise 58, in that it gives equivalent characterization of the modal logic of frames over which R is an equivalence (the logic traditionally referred to as $\mathsf{S5}$).

44. Proposition: Let R be an equivalence relation, and for each $w \in W$ define the *equivalence class* of w as the set $[w] = \{w' \in W : Rww'\}$. Then:

- $w \in [w]$;
- R is universal on each equivalence class $[w]$;
- The collection of equivalence classes partitions W into mutually exclusive and jointly exhaustive subsets.

45. Proposition: A formula φ is valid in all frames $F = (W, R)$ where R is an equivalence relation, if and only if it valid in all frames $F = (W, R)$ where R is universal. Hence, the logic of universal frames is just $\mathsf{S5}$.

Proof. It's immediate to verify that a universal relation R on W is an equivalence. Hence, if φ is valid in all frames where R is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose ψ is a formula that fails at a world w in a model $M = (W, R, V)$ based on a frame (W, R), where R is an equivalence on W. So $M \not\models_w \psi$. Define a model $M' = (W', R', V')$ as follows:

- $W' = [w]$;
- R' is universal on W';
- $V'(p) = V(p) \cap W'$.

(So the set W' of worlds in M' is represented by the shaded area in Figure 5.) It is easy to see that R and R' agree on W'. Then one can show by induction on formulas that for all $w' \in W'$: $M' \models_{w'} \varphi$ if and only if $M \models_{w'} \varphi$ for each φ (this makes sense since $W' \subseteq W$). In particular, $M' \not\models_w \psi$, and ψ fails in a model based on a universal frame. ■

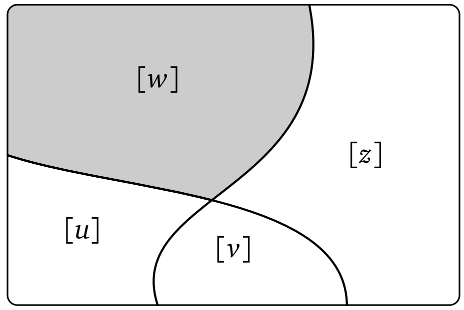


Figure 5: A partition of W in equivalence classes.

Part II

Meta-Theory of Modal Logic

6. Modal systems

46. Definition: A *modal logic* is a set Σ of modal sentences which is closed under *tautological implication* in the following sense: if $\varphi_1, \ldots, \varphi_n \in \Sigma$ and $\varphi_1 \to (\varphi_2 \to \cdots(\varphi_n \to \varphi)\cdots)$ is a tautological instance, then $\varphi \in \Sigma$.

47. Proposition: Every modal logic is closed under the rule of *Modus Ponens*:

$$\text{MP.} \quad \frac{\varphi \to \psi \quad \varphi}{\psi}$$

Proof. $(\varphi \to \psi) \to (\varphi \to \psi)$ is tautological instance, hence if $\varphi \to \psi$ and φ are in Σ, so is ψ. ■

48. Definition: A modal logic Σ is *normal* if it is closed under the rule RK:

$$\text{RK.} \quad \frac{\varphi_1 \to (\varphi_2 \to \cdots(\varphi_{n-1} \to \varphi_n)\cdots)}{\Box\varphi_1 \to (\Box\varphi_2 \to \cdots(\Box\varphi_{n-1} \to \Box\varphi_n)\cdots)}$$

Observe that while tautological implication is “fine-grained” enough to preserve *truth at a world*, the rule RK only preserves *truth in a model* (and hence also validity in a frame or in a class of frames).

49. Proposition: Every normal modal logic Σ is closed under the rule of *Necessitation*:

$$\text{RN.} \quad \frac{\varphi}{\Box\varphi}$$

Proof. RN is just the special case of RK when $n = 1$. ■

50. Proposition: Every normal modal logic Σ contains every instance of K.

Proof. In fact, K follows from rule RK: $(\varphi \to \psi) \to (\varphi \to \psi)$ is in Σ since it is a tautological instance; one application of RK gives that $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is in Σ as well. ∎

51. Exercise: Show that every normal modal logic Σ contains $\neg\Diamond\bot$.

52. Proposition: Let $\mathsf{S}_1, \dots, \mathsf{S}_n$ be schemas. Then there is a smallest modal logic Σ containing all instances of $\mathsf{S}_1, \dots, \mathsf{S}_n$. Such a modal logic is called a *modal system* and denoted by $\mathsf{KS}_1 \dots \mathsf{S}_n$. The smallest normal modal logic is denoted by K.

Proof. Given $\mathsf{S}_1, \dots, \mathsf{S}_n$, define Σ as the intersection of all normal modal logics containing all instances of $\mathsf{S}_1, \dots, \mathsf{S}_n$. The intersection is non-empty as $\mathsf{Fml}_{\mathscr{L}}$, the set of all formulas, is such a modal logic. ∎

53. Definition: Given a modal system $\mathsf{KS}_1 \dots \mathsf{S}_n$ and a formula φ we say that φ is *provable* in $\mathsf{KS}_1 \dots \mathsf{S}_n$, written $\mathsf{KS}_1 \dots \mathsf{S}_n \vdash \varphi$ if and only if there are formulas $\varphi_1, \dots, \varphi_n$ such that $\varphi_n = \varphi$ and each φ_i is either a tautological instance, or an instance of the schemas $\mathsf{K}, \mathsf{S}_1, \dots, \mathsf{S}_n$, or it follows from previous formulas by means of the rules MP or RN.

The following proposition allows us to show that $\varphi \in \Sigma$ by exhibiting a Σ-proof of φ.

54. Proposition: $\mathsf{KS}_1 \dots \mathsf{S}_n = \{\varphi : \mathsf{KS}_1 \dots \mathsf{S}_n \vdash \varphi\}$.

Proof. We use induction on the length of proofs to show that $\{\varphi : \mathsf{KS}_1 \dots \mathsf{S}_n \vdash \varphi\} \subseteq \mathsf{KS}_1 \dots \mathsf{S}_n$. The converse inclusion follows by showing that $\{\varphi : \mathsf{KS}_1 \dots \mathsf{S}_n \vdash \varphi\}$ is a normal modal logic containing all the instances of the schemas $\mathsf{S}_1, \dots, \mathsf{S}_n$, and the observation that $\mathsf{KS}_1 \dots \mathsf{S}_n$ is, by definition, the smallest such logic. ∎

In order to practice proofs in the smallest modal system, we show the valid formulas on the left-hand side of the table of Exercise 28 can all be given K-proofs. Justifications for steps that are either tautological instances or follow by tautological implication from previous one are just marked "PL" (for "Propositional Logic").

$\mathsf{K} \vdash \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$:

1.	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	PL
2.	$\Box[(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	RN
3.	$\Box(\varphi \to \psi) \to \Box(\neg\psi \to \neg\varphi)$	K, MP
4.	$\Box(\neg\psi \to \neg\varphi) \to (\Box\neg\psi \to \Box\neg\varphi)$	K
5.	$\Box(\varphi \to \psi) \to (\Box\neg\psi \to \Box\neg\varphi)$	PL, 3,4
6.	$(\Box\neg\psi \to \Box\neg\varphi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$	PL
7.	$\Box(\varphi \to \psi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$	PL, 5, 6
8.	$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	re-writing.

$\mathsf{K} \vdash \Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$:

1.	$\varphi \to (\neg\psi \to \neg(\varphi \to \psi))$	PL
2.	$\Box[\varphi \to (\neg\psi \to \neg(\varphi \to \psi))]$	RN
3.	$\Box\varphi \to \Box(\neg\psi \to \neg(\varphi \to \psi))$	K
4.	$\Box\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \to \psi))$	K
5.	$\Box\varphi \to (\neg\Box\neg(\varphi \to \psi) \to \neg\Box\neg\psi)$	PL
6.	$\Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$	re-writing.

$\mathsf{K} \vdash \Box(\varphi \,\&\, \psi) \to (\Box\varphi \,\&\, \Box\psi)$:

1.	$\neg(\varphi \to \neg\psi) \to \varphi$	PL
2.	$\Box\neg(\varphi \to \neg\psi) \to \Box\varphi$	RN, K
3.	$\Box(\varphi \,\&\, \psi) \to \Box\varphi$	re-writing
4.	$\Box(\varphi \,\&\, \psi) \to \Box\psi$	similarly
5.	$\Box(\varphi \,\&\, \psi) \to (\Box\varphi \,\&\, \Box\psi)$	PL, 3,4.

$\mathsf{K} \vdash (\Box\varphi \,\&\, \Box\psi) \to \Box(\varphi \,\&\, \psi)$:

1.	$\varphi \to (\psi \to \neg(\varphi \to \neg\psi))$	PL
2.	$\Box\varphi \to (\Box\psi \to \Box\neg(\varphi \to \neg\psi))$	RN, K
3.	$(\Box\varphi \,\&\, \Box\psi) \to \Box(\varphi \,\&\, \psi)$	PL, re-writing.

$\mathsf{K} \vdash \Box\varphi \to \Box(\psi \to \varphi)$:

1.	$\varphi \to (\psi \to \varphi)$	PL
2.	$\Box\varphi \to \Box(\psi \to \varphi)$	RN, K.

$\mathsf{K} \vdash \neg\Diamond\varphi \to \Box(\varphi \to \psi)$:

1.	$\neg\varphi \to (\varphi \to \psi)$	PL
2.	$\Box\neg\varphi \to \Box(\varphi \to \psi)$	RN, K
3.	$\neg\neg\Box\neg\varphi \to \Box(\varphi \to \psi)$	PL
4.	$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$	re-writing.

$\mathsf{K} \vdash (\Diamond\varphi \vee \Diamond\psi) \to \Diamond(\varphi \vee \psi)$:

1.	$\neg(\neg\varphi \to \psi) \to \neg\varphi$	PL
2.	$\neg(\varphi \vee \psi) \to \neg\varphi$	re-writing
3.	$\Box\neg(\varphi \vee \psi) \to \Box\neg\varphi$	RN, K
4.	$\neg\Box\neg\varphi \to \neg\Box\neg(\varphi \vee \psi)$	PL
5.	$\Diamond\varphi \to \Diamond(\varphi \vee \psi)$	re-writing
6.	$\Diamond\psi \to \Diamond(\varphi \vee \psi)$	similarly
7.	$(\Diamond\varphi \vee \Diamond\psi) \to \Diamond(\varphi \vee \psi)$	PL, 5,6.

$\mathsf{K} \vdash \Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi)$:

1.	$\neg\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \vee \psi)$	PL, re-writing
2.	$\Box\neg\varphi \rightarrow (\Box\neg\psi \rightarrow \Box\neg(\varphi \vee \psi)$	RN, K
3.	$\Box\neg\varphi \rightarrow (\neg\Box\neg(\varphi \vee \psi) \rightarrow \neg\Box\neg\psi))$	PL
4.	$\neg\Box\neg(\varphi \vee \psi) \rightarrow (\Box\neg\varphi \rightarrow \neg\Box\neg\psi)$	PL
5.	$\neg\Box\neg(\varphi \vee \psi) \rightarrow (\neg\neg\Box\neg\psi \rightarrow \neg\Box\neg\varphi)$	PL
6.	$\Diamond(\varphi \vee \psi) \rightarrow (\neg\Diamond\psi \rightarrow \Diamond\varphi)$	re-writing
7.	$\Diamond(\varphi \vee \psi) \rightarrow (\Diamond\psi \vee \Diamond\varphi)$	re-writing.

55. Definition: Each of the schemas T, B, $\mathsf{4}$, and $\mathsf{5}$ has a *dual*, denoted by a subscripted diamond, as follows:

$\mathsf{T}_\Diamond$: $\varphi \rightarrow \Diamond\varphi$
$\mathsf{B}_\Diamond$: $\Diamond\Box\varphi \rightarrow \varphi$
$\mathsf{4}_\Diamond$: $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$
$\mathsf{5}_\Diamond$: $\Diamond\Box\varphi \rightarrow \Box\varphi$

Each of the dual above schemas is obtained from the corresponding schema by replacing $\neg\varphi$ for φ, contraposing, and re-writing. Schema D is its own dual (modulo the replacement of $\neg\Diamond\neg$ by $\Box$).

56. Exercise: Show that for each schema S in Definition 55: $\mathsf{K} \vdash \mathsf{S} \leftrightarrow \mathsf{S}_\Diamond$.

We now come to proofs in systems of modal logic other than K.

57. Proposition: The following provability results obtain:

(*a*) $\mathsf{KT5} \vdash \mathsf{B}$;
(*b*) $\mathsf{KT5} \vdash \mathsf{4}$;
(*c*) $\mathsf{KDB4} \vdash \mathsf{T}$;
(*d*) $\mathsf{KB4} \vdash \mathsf{5}$;
(*e*) $\mathsf{KB5} \vdash \mathsf{4}$;
(*f*) $\mathsf{KT} \vdash \mathsf{D}$.

Proof. We exhibit proofs for each.

$\mathsf{KT5} \vdash \mathsf{B}$:

1.	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$	$\mathsf{5}$
2.	$\varphi \rightarrow \Diamond\varphi$	$\mathsf{T}\Diamond$
3.	$\varphi \rightarrow \Box\Diamond\varphi$	PL.

$\mathsf{KT5} \vdash \mathsf{4}$:

1.	$\Diamond\Box\varphi \rightarrow \Box\Diamond\Box\varphi$	$\mathsf{5}$ with $\Box\varphi$ for φ
2.	$\Box\varphi \rightarrow \Diamond\Box\varphi$	$\mathsf{T}_\Diamond$ with $\Box\varphi$ for φ
3.	$\Box\varphi \rightarrow \Box\Diamond\Box\varphi$	PL, 1,2
4.	$\Diamond\Box\varphi \rightarrow \Box\varphi$	$\mathsf{5}_\Diamond$
5.	$\Box\Diamond\Box\varphi \rightarrow \Box\Box\varphi$	RK, 4
6.	$\Box\varphi \rightarrow \Box\Box\varphi$	PL, 3, 5.

$\mathsf{KDB4} \vdash \mathsf{T}$:

1.	$\Diamond\Box\varphi \to \varphi$	$\mathsf{B}_\Diamond$
2.	$\Box\Box\varphi \to \Diamond\Box\varphi$	D with $\Box\varphi$ for φ
3.	$\Box\Box\varphi \to \varphi$	PL 1, 2
4.	$\Box\varphi \to \Box\Box\varphi$	$\mathsf{4}$
5.	$\Box\varphi \to \varphi$	PL, 1,4.

$\mathsf{KB4} \vdash \mathsf{5}$:

1.	$\Diamond\varphi \to \Box\Diamond\Diamond\varphi$	B with $\Diamond\varphi$ for φ
2.	$\Diamond\Diamond\varphi \to \Diamond\varphi$	$\mathsf{4}_\Diamond$
3.	$\Box\Diamond\Diamond\varphi \to \Box\Diamond\varphi$	RK, 2
4.	$\Diamond\varphi \to \Box\Diamond\varphi$	PL, 1,3.

$\mathsf{KB5} \vdash \mathsf{4}$:

1.	$\Box\varphi \to \Box\Diamond\Box\varphi$	B with $\Box\varphi$ for φ
2.	$\Diamond\Box\varphi \to \Box\varphi$	$\mathsf{5}_\Diamond$
3.	$\Box\Diamond\Box\varphi \to \Box\Box\varphi$	RK, 2
4.	$\Box\varphi \to \Box\Box\varphi$	PL, 1,3.

$\mathsf{KT} \vdash \mathsf{D}$:

1.	$\Box\varphi \to \varphi$	T
2.	$\varphi \to \Diamond\varphi$	$\mathsf{T}_\Diamond$
3.	$\Box\varphi \to \Diamond\varphi$	PL, 1, 2

■

58. Exercise: Show that $\mathsf{KTB4} = \mathsf{KT5} = \mathsf{KDB4} = \mathsf{KDB5}$.

59. Definition: Following tradition, we define $\mathsf{S4}$ to be the system $\mathsf{KT4}$, and $\mathsf{S5}$ the system $\mathsf{KTB4}$.

Exercise 58 shows that the classical system $\mathsf{S5}$ has several equivalent axiomatizations (see Exercise 43).

7. Soundness and its consequences

In the previous section we saw how to prove that two systems of modal logic are in fact the same system. Theorem 60 gives us a tool to show that two such systems are distinct.

60. Soundness Theorem: If schemas $\mathsf{S}_1, \dots, \mathsf{S}_n$ are valid in the classes of models $\mathscr{C}_n, \dots, \mathscr{C}_n$, respectively, then $\mathsf{KS}_1 \dots \mathsf{S}_n \vdash \varphi$ implies that φ is valid in the class of models $\mathscr{C}_n \cap \dots \cap \mathscr{C}_n$.

Proof. By induction on length of proofs. For brevity, put $\mathscr{C} = \mathscr{C}_n \cap \dots \cap \mathscr{C}_n$.

Basis: If φ has a proof of length 1, then it is either a tautological instance or an instance of K, or an instance of one of the schemas. In the first case, φ is valid in $\mathscr{C}$, since tautological instance are valid in *any* class of models, by Corollary 20. Similarly in the second case, by Theorem 25. Finally in the third case, since φ is valid in $\mathscr{C}_i$ and $\mathscr{C} \subseteq \mathscr{C}_i$, we have that φ is valid in $\mathscr{C}$ as well.

Inductive step: Suppose φ has a proof of length $k > 1$. If φ is a tautological instance or an instance of one of the schemas, we proceed as in the previous step. So suppose φ is obtained by MP from previous formulas $\psi \to \varphi$ and ψ. Then $\varphi \to \psi$ and ψ have proofs of length $< k$, and by inductive hypothesis they are valid in $\mathscr{C}$. By Proposition 26, φ is valid in $\mathscr{C}$ as well. Finally suppose φ is obtained by RN from ψ (so that $\varphi = \Box\psi$). By inductive hypothesis, ψ is valid in $\mathscr{C}$, and by Proposition 21 so is φ. ■

The Soundness theorem is useful because it allows us to show that two modal systems Σ and Σ' are distinct, by finding a formula φ such that $\Sigma' \vdash \varphi$ that fails in a model of Σ.

61. Proposition: $\mathsf{KD} \subsetneq \mathsf{KT}$

Proof. This is the syntactic counterpart to the semantic fact that all reflexive relations are serial. To show $\mathsf{KD} \subseteq \mathsf{KT}$ we need to see that $\mathsf{KD} \vdash \varphi$ implies $\mathsf{KT} \vdash \varphi$, which follows from $\mathsf{KT} \vdash \mathsf{D}$, as shown in Proposition 57, part (f). To show that the inclusion is proper, by Soundness (Theorem 60), it suffices to exhibit a model of KD where some instance $\Box\varphi \to \varphi$ of T fails (an easy task left to the reader), for then by Soundness $\mathsf{KD} \nvdash \Box\varphi \to \varphi$. ■

62. Proposition: $\mathsf{KB} \neq \mathsf{K4}$.

Proof. We construct a symmetric model where some instance of 4 fails; since obviously the instance is provable for $\mathsf{K4}$ but not in KB, it will follow $\mathsf{K4} \nsubseteq \mathsf{KB}$. Consider the symmetric model M of Figure 6 Since the model is symmetric, K and B are true in M (by Theorems 25 and 36, respectively). However, $M \nvDash_w \Box p \to \Box\Box p$. ■

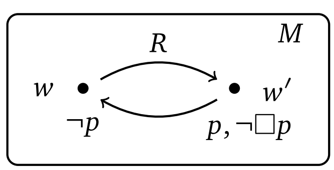


Figure 6: A symmetric model falsifying an instance of 4.

63. Theorem: $\mathsf{KTB} \nvdash 4$ and $\mathsf{KTB} \nvdash 5$.

Proof. By Theorem 36 we know that all instances of T and B are true in each reflexive symmetric model (respectively). So by Soundness it suffices to find a reflexive symmetric model containing a world at which some instance of 4 fails, and similarly for 5. We use the same model for both claims. Consider the symmetric, reflexive model in Figure 7. Then $M \not\models_{w_1} \Box p \to \Box\Box p$, so the instance of 4 with $\varphi = p$ fails at w_1. Similarly, $M \not\models_{w_2} \Diamond\neg p \to \Box\Diamond\neg p$, so the instance of 5 with $\varphi = \neg p$ fails at w_2. ■

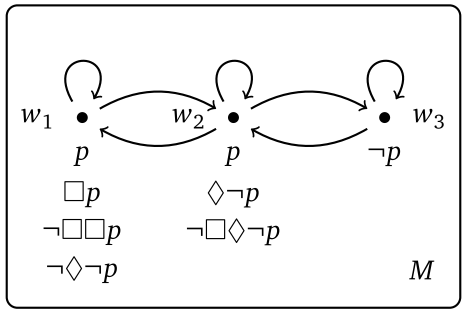


Figure 7: The model for Theorem 63.

64. Theorem: $\mathsf{KD5} \neq \mathsf{KT4} = \mathsf{S4}$.

Proof. By Theorem 36 we know that all instances of D and 5 to be true in all serial euclidean models. So it suffices to find a serial euclidean model containing a world at which some instance of 4 fails. Consider the model of Figure 8, and notice that $M \not\models_{w_1} \Box p \to \Box\Box p$. ■

65. Exercise: Give an alternative proof of Theorem 64 using a model with 3 worlds.

66. Exercise: Provide a single reflexive transitive model showing that both $\mathsf{KT4} \nvdash \mathsf{B}$ and $\mathsf{KT4} \nvdash 5$.

8. Maximally consistent sets

In Section 6 we defined a notion of provability of a formula in a system Σ. We now extend this notion to provability in Σ from formulas in a set Γ.

67. Definition: A formula φ is provable in a system Σ from a set of formulas Γ, written $\Gamma \vdash_\Sigma \varphi$ if and only if there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$.

68. Proposition: Let Σ be a modal system and Γ a set of modal formulas. The following properties hold:

(*a*) *Monotony*: If $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_\Sigma \varphi$;
(*b*) *Reflexivity*: If $\varphi \in \Gamma$ then $\Gamma \vdash_\Sigma \varphi$;
(*c*) *Cut*: If $\Gamma \vdash_\Sigma \varphi$ and $\Delta \cup \{\varphi\} \vdash_\Sigma \psi$ then $\Gamma \cup \Delta \vdash_\Sigma \psi$;
(*d*) *Deduction theorem*: $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi$ if and only if $\Gamma \vdash_\Sigma \psi \to \varphi$;
(*e*) *Rule T*: If $\Gamma \vdash_\Sigma \varphi_1$ and ...and $\Gamma \vdash_\Sigma \varphi_n$ and $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \psi) \cdots)$ is a tautological instance, then $\Gamma \vdash_\Sigma \psi$.

The proof is an easy exercise. Part (*e*) of Proposition 68 gives us that, for instance, if $\Gamma \vdash_\Sigma \varphi \vee \psi$ and $\Gamma \vdash_\Sigma \neg\varphi$, then $\Gamma \vdash_\Sigma \psi$. Also, in what follows, we write $\Gamma, \varphi \vdash_\Sigma \psi$ instead of $\Gamma \cup \{\varphi\} \vdash_\Sigma \psi$.

69. Definition: A set Γ is *deductively closed* relatively to a system Σ if and only if $\Gamma \vdash_\Sigma \varphi$ implies $\varphi \in \Gamma$.

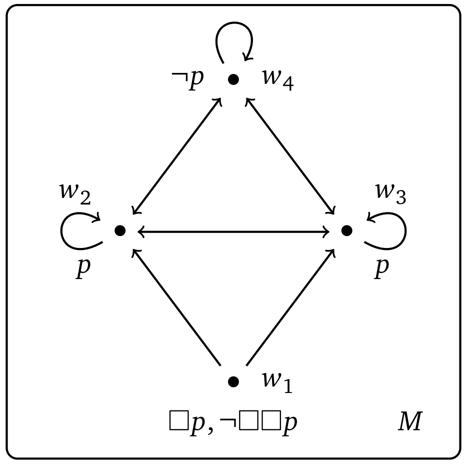


Figure 8: The model for Theorem 64.

70. Definition: A set Γ is *consistent* relatively to a system Σ or, as we wil say, Σ-consistent, if and only if $\Gamma \nvdash_\Sigma \bot$.

So for instance, the set $\{\Box(p \to q), \Box p, \neg\Box q\}$ is consistent relatively to propositional logic, but not K-consistent. Similarly, the set $\{\Diamond p, \Box\Diamond p \to q, \neg q\}$ is not $\mathsf{K5}$-consistent.

71. Proposition: Let Γ be a set of formulas. Then:

(a) A set Γ is Σ-consistent if and only if there is some formula φ such that $\Gamma \nvdash_\Sigma \varphi$.

(b) $\Gamma \vdash_\Sigma \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is not Σ-consistent.

(c) If Γ is Σ-consistent, then for any formula φ, either $\Gamma \cup \{\varphi\}$ is Σ-consistent or $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent.

Proof. These fact follow easily using classical propositional logic. We give the argument for (c). Proceed contrapositively and suppose neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent. Then by (b) both $\Gamma, \varphi \vdash_\Sigma \bot$ and $\Gamma, \neg\varphi \vdash_\Sigma \bot$. By the deduction theorem $\Gamma \vdash_\Sigma \varphi \to \bot$ and $\Gamma \vdash_\Sigma \neg\varphi \to \bot$. But $(\varphi \to \bot) \to ((\neg\varphi \to \bot) \to \bot)$ is a tautological instance, hence by Proposition 68, part (e), $\Gamma \vdash_\Sigma \bot$. ■

72. Definition: A set Γ is *maximally* Σ*-consistent* if and only if it is Σ-consistent and for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

73. Proposition: Suppose Γ is maximally consistent in Σ. Then:

(a) Γ is deductively closed in Σ.

(b) $\Sigma \subseteq \Gamma$.

(c) $\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.

(d) $\varphi \to \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ implies $\psi \in \Gamma$.

Proof. For part (a): if $\Gamma \vdash_\Sigma \varphi$ but $\varphi \notin \Gamma$ then by maximality $\neg\varphi \in \Gamma$, and Γ is inconsistent. For part (b): if $\varphi \in \Sigma$ then $\Gamma \vdash_\Sigma \varphi$, and $\varphi \in \Gamma$ by deductive closure. For part (c): if $\neg\varphi \in \Gamma$, then by consistency

$\varphi \notin \Gamma$; and if $\varphi \notin \Gamma$ then by maximality $\varphi \in \Gamma$. Finally, for part (d): suppose $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$; then $\Gamma \vdash_\Sigma \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \to \psi \notin \Gamma$ then by maximality $\neg(\varphi \to \psi) \in \Gamma$, so by Rule T, deductive closure, and consistency both $\varphi \in \Gamma$ and $\psi \notin \Gamma$. ■

74. Theorem: (*Lindenbaum's Lemma*) If Γ is Σ-consistent then there is a maximally Σ-consistent set Δ extending Γ.

Proof. Let $\varphi_0, \varphi_1, \ldots$ be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing p_0, and at each stage n list the finitely many formulas of length n using only variables among $p_0, \ldots, p_n$. We define sets of formulas Δ_n by recursion on n, and we then set $\Delta = \lim_n \Delta_n$. We first put $\Delta_0 = \Gamma$, then supposing that Δ_n has been defined:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

If we now let $\Delta = \bigcup_n \Delta_n$, we can show the following:

(a) For each n, $\Delta_n \subseteq \Delta$ (immediate from the definition).

(b) $\Gamma \subseteq \Delta$ (from (a)).

(c) If $n \leq m$ then $\Delta_n \subseteq \Delta_m$ (by induction on $m - n$).

(d) Δ is maximal (by construction).

(e) For each m, Δ_m is consistent (by induction on m, using Proposition 71, part (c))

(f) If $\Delta' \subseteq \Delta$ is finite, then there is m such that $\Delta' \subseteq \Delta_m$.

(g) Δ is consistent.

It follows that Δ is a maximally Σ-consistent set extending Γ. ■

75. Corollary: $\Gamma \vdash_\Sigma \varphi$ if and only if $\varphi \in \Delta$ for each maximally Σ-consistent set Δ extending Γ (including when $\Gamma = \varnothing$, in which case we get another characterization of the modal system Σ.)

Proof. Suppose $\Gamma \vdash_\Sigma \varphi$, and let Δ be any maximally Σ-consistent set extending Γ. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so $\Delta \vdash_\Sigma \varphi$ (by monotony) and $\Delta \vdash_\Sigma \neg\varphi$ (by reflexivity), and so Δ is inconsistent. Conversely if $\Gamma \nvdash_\Sigma \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent, and by Lindenbaum's Lemma there is a maximally consistent set Δ extending $\Gamma \cup \{\neg\varphi\}$; by consistency, $\varphi \notin \Delta$. ■

When, in the next section, we construct a model whose set of worlds is given by the maximally consistent sets in some normal modal logic Σ, we will also need to define an accessibility relation between such "worlds." The next few lemmas give us the tools to do so. As noted, Σ will be a normal modal logic throughout.

76. Lemma: If $\Gamma \vdash_\Sigma \varphi$ then $\{\Box\psi : \psi \in \Gamma\} \vdash \Box\varphi$.

Proof. If $\Gamma \vdash_\Sigma \varphi$ then there are $\psi_1, \ldots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$. Since Σ is normal, by rule RK, $\Sigma \vdash \Box\psi_1 \to (\Box\psi_2 \to \cdots (\Box\psi_n \to \Box\varphi) \cdots)$, where obviously $\Box\psi_1, \ldots, \Box\psi_k \in \{\Box\psi : \psi \in \Gamma\}$. Hence, by definition, $\{\Box\psi : \psi \in \Gamma\} \vdash \Box\varphi$. ■

77. Lemma: If $\{\psi : \Box\psi \in \Gamma\} \vdash_\Sigma \varphi$ then $\Gamma \vdash_\Sigma \Box\varphi$.

Proof. Let $\Delta = \{\psi : \Box\psi \in \Gamma\}$, so that $\Delta \vdash_\Sigma \varphi$; then by Lemma 76, $\{\Box\psi : \psi \in \Delta\} \vdash \Box\varphi$. But obviously $\{\Box\psi : \psi \in \Delta\} \subseteq \Gamma$, so also $\Gamma \vdash_\Sigma \Box\varphi$ by Monotony. ■

78. Theorem: If Γ is maximally Σ-consistent, then $\Box\varphi \in \Gamma$ if and only if for every maximally Σ-consistent Δ such that $\{\psi : \Box\psi \in \Gamma\} \subseteq \Delta$, it holds that $\varphi \in \Delta$.

Proof. The left-to-right half of the theorem is obvious. For the converse, suppose $\Box\varphi \notin \Gamma$. Since Γ is deductively closed, $\Gamma \nvdash_\Sigma \Box\varphi$, and by Lemma 77 $\{\psi : \Box\psi \in \Gamma\} \nvdash_\Sigma \varphi$. By Proposition 71, part (b), $\{\psi : \Box\psi \in \Gamma\} \cup \{\neg\varphi\}$ is Σ-consistent, so that by Lindenbaum's Lemma there is a maximally Σ-consistent set Δ such that $\{\psi : \Box\psi \in \Gamma\} \cup \{\neg\varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$, and the theorem is proved. ■

79. Lemma: Suppose Γ and Δ are maximally Σ-consistent. Then: $\{\varphi : \Box\varphi \in \Gamma\} \subseteq \Delta$ if and only if $\{\Diamond\varphi : \varphi \in \Delta\} \subseteq \Gamma$.

Proof. "Only if" direction: Assume $\{\varphi : \Box\varphi \in \Gamma\} \subseteq \Delta$ and suppose $\varphi \in \Delta$. In order to show $\Diamond\varphi \in \Gamma$ suffices to show $\Box\neg\varphi \notin \Gamma$ for then by maximality $\neg\Box\neg\varphi \in \Gamma$. Now, if $\Box\neg\varphi \in \Gamma$ then by hypothesis $\neg\varphi \in \Delta$, against the consistency of Δ (since $\varphi \in \Delta$). Hence $\Box\neg\varphi \notin \Gamma$, as required.

"If" direction: Assume $\{\Diamond\varphi : \varphi \in \Delta\} \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box\varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so by hypothesis $\Diamond\neg\varphi \in \Gamma$. But in a normal modal logic $\Diamond\neg\varphi$ is equivalent to $\neg\Box\varphi$, and if the latter is in Γ by consistency $\Box\varphi \notin \Gamma$, as required. ■

80. Corollary: If Γ is maximally Σ-consistent, then $\Diamond\varphi \in \Gamma$ if and only if for some maximally Σ-consistent Δ such that $\{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma$, it holds that $\varphi \in \Delta$.

Proof. Suppose Γ is maximally Σ-consistent, and argue as follows:

$$
\begin{array}{lcll}
\Diamond\varphi \in \Gamma & \Leftrightarrow & \neg\Box\neg\varphi \in \Gamma, & \text{re-writing;} \\
& \Leftrightarrow & \Box\neg\varphi \notin \Gamma, & \Gamma \text{ is max. } \Sigma\text{-cons.;} \\
& \Leftrightarrow & \exists\Delta\left[\Delta \text{ is max. } \Sigma\text{-cons.} \,\&\, \{\psi : \Box\psi \in \Gamma\} \subseteq \Delta \,\&\, \neg\varphi \notin \Delta\right], & \text{Theorem 78;} \\
& \Leftrightarrow & \exists\Delta\left[\Delta \text{ is max. } \Sigma\text{-cons.} \,\&\, \{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma \,\&\, \neg\varphi \notin \Delta\right], & \text{Lemma 79;} \\
& \Leftrightarrow & \exists\Delta\left[\Delta \text{ is max. } \Sigma\text{-cons.} \,\&\, \{\Diamond\psi : \psi \in \Delta\} \subseteq \Gamma \,\&\, \varphi \in \Delta\right], & \Delta \text{ is maximal.}
\end{array}
$$
■

9. Completeness via canonical models

81. Definition: A model M is said to *determine* a normal modal logic Σ precisely when $M \models \varphi$ if and only if $\Sigma \vdash \varphi$, for all formulas φ.

82. Definition: Let Σ be a normal modal logic. The *canonical model* for Σ is $M^\Sigma = (W^\Sigma, R^\Sigma, V^\Sigma)$, where:

- ▹ $M^\Sigma = \{w \subseteq \mathsf{Fml}_{\mathcal{L}} : w \text{ is maximally } \Sigma\text{-consistent}\}$.
- ▹ $R^\Sigma ww'$ holds if and only if $\{\varphi : \Box\varphi \in w\} \subseteq w'$.
- ▹ $V^\Sigma(p) = \{w : p \in w\}$.

83. Proposition: (*Truth Lemma*) For every formula φ, $M^\Sigma \models_w \varphi$ if and only if $\varphi \in w$.

Proof. By induction on φ. *Basis*: if φ is a propositional variable, say p, then:

$$M^\Sigma \models_w p \Longleftrightarrow w \in V^\Sigma(p) \Longleftrightarrow p \in w.$$

If φ is $\bot$ then both $M^\Sigma \not\models_w \bot$ and $\bot \notin w$ (by consistency of w). The cases for $\neg\varphi$ and $\varphi \to \psi$ follow from the inductive hypothesis and Proposition 73, parts (c) and (d). Here is the case for $\Box\varphi$; in one direction:

$$\begin{array}{llll} M^\Sigma \models_w \Box\varphi & \Rightarrow & \forall w' \in W^\Sigma (R^\Sigma ww' \Rightarrow M^\Sigma \models_{w'} \varphi), & \text{def. } \models; \\ & \Rightarrow & \forall w' \in W^\Sigma (\{\psi : \Box\psi \in w\} \subseteq w' \Rightarrow M^\Sigma \models_{w'} \varphi), & \text{def. } R^\Sigma; \\ & \Rightarrow & \forall w' \in W^\Sigma (\{\psi : \Box\psi \in w\} \subseteq w' \Rightarrow \varphi \in w'), & \text{ind. hyp.}; \\ & \Rightarrow & \Box\varphi \in w, & \text{Theorem 78.} \end{array}$$

Conversely, assume $\Box\varphi \in w$, and let w' be an arbitrary world in W^Σ such that $R^\Sigma ww'$. By definition of R^Σ, we have $\{\psi : \Box\psi \in w\} \subseteq w'$, which immediately gives $\varphi \in w'$. By induction hypothesis, $M^\Sigma \models_{w'} \varphi$, and since w' was arbitrary, $M^\Sigma \models_w \Box\varphi$. ■

84. Theorem: (*Determination*) For every normal modal logic Σ: $M^\Sigma \models \varphi$ if and only if $\Sigma \vdash \varphi$.

Proof. If $M^\Sigma \models \varphi$, then for every maximally Σ-consistent w, we have $M^\Sigma \models_w \varphi$. Hence, by the Truth Lemma, $\varphi \in w$ for every maximally Σ-consistent w, whence by Theorem 75 (with $\Gamma = \varnothing$), $\Sigma \vdash \varphi$. Conversely, if $\Sigma \vdash \varphi$ then by Proposition 73, part (b), every maximally Σ-consistent w contains φ, and hence by the Truth Lemma $M^\Sigma \models_w \varphi$ for every w, i.e., $M^\Sigma \models \varphi$. ■

85. Corollary: The basic modal logic K is complete with respect to the class of all models, i.e., if $\models \varphi$ then $\mathsf{K} \vdash \varphi$.

Proof. Contrapositively, if $\mathsf{K} \nvdash \varphi$ then by Determination $M^{\mathsf{K}} \not\models \varphi$ and hence φ is not valid. ■

Similar results can be extended to other modal system, once we show that the canonical model for a given logic has the corresponding frame property.

86. Theorem: If a normal modal logic Σ contains one of the schemas on the left-hand side of the table of Figure 9, then the canonical model for Σ has the corresponding property on the right-hand side.

If Σ contains . . .	*. . . the canonical model for Σ is:*
D: $\Box\varphi \to \Diamond\varphi$	serial;
T: $\Box\varphi \to \varphi$	reflexive;
B: $\varphi \to \Box\Diamond\varphi$	symmetric;
4: $\Box\varphi \to \Box\Box\varphi$	transitive;
5: $\Diamond\varphi \to \Box\Diamond\varphi$	euclidean.

Figure 9: Basic correspondence facts.

Proof. We take each of these up in turn. Suppose Σ contains D, and let $w \in W^{\Sigma}$; we need to show that there is a w' such that $R^{\Sigma}ww'$. It suffices to show that $\{\psi : \Box\psi \in w\}$ is Σ-consistent, for then by Lindenbaum's Lemma, there is a maximally Σ-consistent set $w' \supseteq \{\psi : \Box\psi \in w\}$, and by definition of R^{Σ} we have $R^{\Sigma}ww'$. So, suppose for contradiction that $\{\psi : \Box\psi \in w\}$ is *not* Σ-consistent, i.e., $\{\psi : \Box\psi \in w\} \vdash_{\Sigma} \bot$. By Lemma 77, $w \vdash_{\Sigma} \Box\bot$, and since Σ contains D, also $w \vdash_{\Sigma} \Diamond\bot$. But Σ is normal, so $\Sigma \vdash \neg\Diamond\bot$ (Exercise 51), whence also $w \vdash_{\Sigma} \neg\Diamond\bot$, against the consistency of w.

Now suppose Σ contains T, and let $w \in W^{\Sigma}$. We want to show $R^{\Sigma}ww$, i.e., $\{\varphi : \Box\varphi \in w\} \subseteq w$. But if $\Box\varphi \in w$ then by T also $\varphi \in w$, as desired.

Now suppose Σ contains B, and suppose $R^{\Sigma}uv$ for $u, v \in W^{\Sigma}$. We need to show that $R^{\Sigma}vu$, i.e., $\{\varphi : \Box\varphi \in v\} \subseteq u$. By Lemma 79, this is equivalent to $\{\Diamond\varphi : \varphi \in u\} \subseteq v$. So suppose $\varphi \in u$. By B, also $\Box\Diamond\varphi \in u$. By the hypothesis that $R^{\Sigma}uv$, we have that $\{\psi : \Box\psi \in u\} \subseteq v$, and hence $\Diamond\varphi \in v$, as required.

Now suppose Σ contains 4, and suppose $R^{\Sigma}uv$ and $R^{\Sigma}vw$. We need to show $R^{\Sigma}uw$. From the hypothesis we have both $\{\psi : \Box\psi \in u\} \subseteq v$ and $\{\psi : \Box\psi \in v\} \subseteq w$. In order to show $R^{\Sigma}uw$ it suffices to show $\{\psi : \Box\psi \in u\} \subseteq w$. So let $\Box\psi \in u$; by 4, also $\Box\Box\psi \in u$ and by hypothesis we get, first, that $\Box\psi \in v$ and, second, that $\psi \in w$, as desired.

Now suppose Σ contains 5, suppose $R^{\Sigma}uv$ and $R^{\Sigma}uw$. We need to show $R^{\Sigma}vw$. The first hypothesis give $\{\varphi : \Box\varphi \in u\} \subseteq v$, and the second hypothesis is equivalent to $\{\Diamond\varphi : \varphi \in w\} \subseteq u$, by Lemma 79. To show $R^{\Sigma}vw$, by Lemma 79, it suffices to show $\{\Diamond\varphi : \varphi \in w\} \subseteq v$. So let $\varphi \in w$; by the second hypothesis $\Diamond\varphi \in u$ and by 5, $\Box\Diamond\varphi \in u$ as well. But now the first hypothesis give $\Diamond\varphi \in v$, as desired. ■

As a corollary we obtain completeness results for a number of systems. For instance, we know that S5 = KT5 = KTB4 is complete with respect to the class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

87. Theorem: Let $\mathscr{C}_{\mathsf{D}}$, $\mathscr{C}_{\mathsf{T}}$, $\mathscr{C}_{\mathsf{B}}$, $\mathscr{C}_{4}$, and $\mathscr{C}_{5}$ be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\mathsf{S}_1, \ldots, \mathsf{S}_n$ among D, T, B, 4, and 5, the system $\mathsf{KS}_1, \ldots, \mathsf{S}_n$ is determined by the class of models $\mathscr{C} = \mathscr{C}_{\mathsf{S}_1} \cap \ldots \cap \mathscr{C}_{\mathsf{S}_n}$.

88. Proposition: Let Σ be a normal modal logic; then:

(a) If Σ contains the schema $\Diamond\varphi \to \Box\varphi$ then the canonical model for Σ is partially functional.

(b) If Σ contains the schema $\Diamond\varphi \leftrightarrow \Box\varphi$ then the canonical model for Σ is functional.

(c) If Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ then the canonical model for Σ is weakly dense.

(see Figure 4 on p. 10 for definitions of these frame properties).

Proof. Notice that Part (b) follows immediately from part (a) and the seriality proof in Theorem 86. So we do parts (a) and (c) in turn.

For (a), suppose that Σ contains the schema $\Diamond\varphi \to \Box\varphi$, to show that R^Σ is partialy functional we need to prove that for any $u, v, w \in W^\Sigma$, if $R^\Sigma wu$ and $R^\Sigma wv$ then $u = v$. Since $R^\Sigma wu$ we have $\{\varphi : \Box\varphi \in w\} \subseteq u$ and since $R^\Sigma wv$ also $\{\varphi : \Box\varphi \in w\} \subseteq v$. The identity $u = v$ will follow if we can establish the two inclusions $u \subseteq v$ and $v \subseteq u$. For the first inclusion, let $\varphi \in u$; then $\Diamond\varphi \in w$, and by the schema and deductive closure of w also $\Box\varphi \in w$, whence by the hypothesis that $R^\Sigma wv$, $\varphi \in v$. The second inclusion is similar, so this establishes part (a).

For part (c): Suppose Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ and to show that R^Σ is weakly dense, let $R^\Sigma uv$. We need to show that there is a maximally Σ-consistent set w such that $R^\Sigma uw$ and $R^\Sigma wv$. Let:

$$\Gamma = \{\varphi : \Box\varphi \in u\} \cup \{\Diamond\psi : \psi \in v\}.$$

It suffices to show that Γ is Σ consistent, for then by Lindenbaum's lemma it can be extendend to a maximally Σ-consistent set w such that $\{\varphi : \Box\varphi \in u\} \subseteq w$ and $\{\Diamond\psi : \psi \in v\} \subseteq w$, i.e., $R^\Sigma uw$ and $R^\Sigma wv$.

Suppose for contradiction that Γ is not consistent. Then there are formulas $\Box\varphi_1, \ldots, \Box\varphi_n \in u$ and $\psi_1, \ldots, \psi_m \in v$ such that $\varphi_1, \ldots, \varphi_n, \Diamond\psi_1, \ldots, \Diamond\psi_m \vdash_\Sigma \bot$. Since $\Diamond(\psi_1 \wedge \ldots \wedge \psi_m) \to (\Diamond\psi_1 \wedge \ldots \ldots \Diamond\psi_m)$ is provable in every normal modal logic, we argue as follows, contradicting the consistency of v:

$$\begin{array}{rlll}
\varphi_1, \ldots, \varphi_n, \Diamond\psi_1, \ldots, \Diamond\psi_m \vdash_\Sigma \bot & \Rightarrow & \varphi_1, \ldots, \varphi_n \vdash_\Sigma (\Diamond\psi_1 \wedge \ldots \wedge \Diamond\psi_m) \to \bot, & \text{Ded. Thm.;} \\
& \Rightarrow & \varphi_1, \ldots, \varphi_n \vdash_\Sigma \Diamond(\psi_1 \wedge \ldots \wedge \psi_m) \to \bot, & \Sigma \text{ is normal;} \\
& \Rightarrow & \varphi_1, \ldots, \varphi_n \vdash_\Sigma \Box\neg(\psi_1 \wedge \ldots \wedge \psi_m), & \text{PL, re-writing;} \\
& \Rightarrow & \Box\varphi_1, \ldots, \Box\varphi_n \vdash_\Sigma \Box\Box\neg(\psi_1 \wedge \ldots \wedge \psi_m), & \text{Lemma 76;} \\
& \Rightarrow & \Box\varphi_1, \ldots, \Box\varphi_n \vdash_\Sigma \Box\neg(\psi_1 \wedge \ldots \wedge \psi_m), & \text{by the schema;} \\
& \Rightarrow & u \vdash_\Sigma \Box\neg(\psi_1 \wedge \ldots \wedge \psi_m), & \text{Monotony;} \\
& \Rightarrow & \Box\neg(\psi_1 \wedge \ldots \wedge \psi_m) \in u, & \text{Ded. closure;} \\
& \Rightarrow & \neg(\psi_1 \wedge \ldots \wedge \psi_m) \in v, & \text{since } R^\Sigma uv.
\end{array}$$

■

On the strength of these examples, one might think that every system Σ of modal logic is *complete*, in the sense that it proves every formula which is valid in every frame in which every theorem of Σ is valid. Unfortunately, there are many systems that are not complete in this sense.

10. The finite model property

The purpose of this section is to establish the decidability of our systems of modal logic by showing that they have the *finite model property*, i.e., that any formula that is true (false) in a model is also true (false) in a *finite* model. The main tool will be that of *filtrations*.

89. Definition: A set Γ of formulas is *closed under subformulas* if it contains every subformula of a formula in Γ. Further, Γ is *modally closed* if it is closed under subformulas and moreover $\varphi \in \Gamma$ implies $\Box\varphi, \Diamond\varphi \in \Gamma$.

90. Definition: Let $M = (W, R, V)$ and suppose Γ is closed under subformulas. Define a relation $\equiv$ on W to hold of any two worlds that make true the same formulas from Γ, i.e.:

$$u \equiv v \quad \text{if and only if} \quad \forall\varphi \in \Gamma : M \models_u \varphi \Leftrightarrow N \models_v \varphi.$$

Clearly, $\equiv$ is an equivalence relation over W. Standardly, for any $w \in W$, the equivalence class of w is denoted by $[w]$.

91. Definition: Let Γ be closed under subformulas and $M = (W, R, V)$. A *filtration of M through* Γ is any model $M^* = (W^*, R^*, V^*)$, where:

(*a*) $W^* = \{[w] : w \in W\}$;

(*b*) For any $u, v \in W$:

- (*i*) If Ruv then $R^*[u][v]$;
- (*ii*) If $R^*[u][v]$ then for any $\Box\varphi \in \Gamma$, if $M \models_u \Box\varphi$ then $M \models_v \varphi$;
- (*iii*) If $R^*[u][v]$ then for any $\Diamond\varphi \in \Gamma$, if $M \models_v \varphi$ then $M \models_u \Diamond\varphi$.

(*c*) $V^*(p) = \{[u] : u \in V(p)\}$.

92. Theorem: If M^* is a filtration of M through Γ, then for every $\varphi \in \Gamma$ and $w \in W$, we have $M \models_w \varphi$ if and only if $M^* \models_{[w]} \varphi$.

Proof. By induction on φ, using the fact that Γ is closed under subformulas. For φ atomic, say p: the left-to-right direction is immediate, as $M \models_w p$ only if $w \in V(p)$, which implies $[w] \in V^*(p)$, i.e., $M^* \models_{[w]} p$. Conversely, suppose $M^* \models_{[w]} p$, i.e., $[w] \in V^*(p)$; then $w \equiv w' \in V(p)$, and since $p \in \Gamma$, also $w \in V(p)$, so that $M \models_w p$. The cases for the Boolean connectives follow immediately from the inductive hypothesis and closure of Γ under subformulas.

So we do the case for $\Box\varphi \in \Gamma$. Suppose $M \models_u \Box\varphi$; to show that $M^* \models_{[u]} \varphi$, let v be such that $R^*[u][v]$. From Definition 91, part (ii), we have that $M \models_v \varphi$, and by inductive hypothesis $M^* \models_{[v]} \varphi$. Since v was arbitrary, $M^* \models_{[u]} \Box\varphi$ follows. Conversely, suppose $M^* \models_{[u]} \Box\varphi$ and let v be arbitrary such that Ruv. From Definition 91, part (i), we have $R^*[u][v]$, so that $M^* \models_{[v]} \varphi$; by inductive hypothesis $M \models_v \varphi$, and since v was arbitrary, $M \models_u \Box\varphi$. ■

93. Corollary: Let Γ be closed under subformulas. Then:

- ▷ If M^* is a filtration of M through Γ then for any $\varphi \in \Gamma$: $M \models \varphi$ if and only if $M^* \models \varphi$.
- ▷ If $\mathscr{C}$ is a class of models and $\Gamma(\mathscr{C})$ is the class of Γ-filtrations of models in $\mathscr{C}$, then any formula $\varphi \in \Gamma$ is valid in $\mathscr{C}$ if and only if it is valid in $\Gamma(\mathscr{C})$.

We have not yet shown that there are any filtrations. But indeed, for any model M, there are many filtrations of M through Γ. We identify two, in particular: the finest and coarsest filtrations. Filtrations of the same models will differ in their accessibility relation (as Definition 91 stipulates directly what W^* and V^* should be like). The finest filtration will have as few related worlds as possible, whereas the coarsest will have as many as possible.

94. Definition: Where Γ is closed under subformulas, the *finest* filtration M^* of a model M is defined by putting:

$$R^*[u][v] \quad \text{if and only if} \quad \exists u' \in [u]\, \exists v' \in [v] : Ru'v'.$$

95. Proposition: The finest filtration M^* is indeed a filtration.

Proof. We need to check that R^*, so defined, satisfies Definition 91, part (b). We check the three conditions in turn.

(i) If Ruv then by reflexivity of $\equiv$, also $R^*[u][v]$.

(ii) Suppose $\Box\varphi \in \Gamma$, $R^*[u][v]$, and $M \models_u \Box\varphi$. By definition of R^*, there are $u' \equiv u$ an $v' \equiv v$ such that $Ru'v'$. Since u and u' agree on Γ, also $M \models_{u'} \Box\varphi$, so that $M \models_{v'} \varphi$. By closure of Γ, v and v' agree on φ, so $M \models_v \varphi$, as desired.

(iii) Suppose $\Diamond\varphi \in \Gamma$, $R^*[u][v]$, and $M \models_v \varphi$. Arguing similarly to the previous case, $M \models_u \Diamond\varphi$. ■

96. Definition: Where Γ is closed under subformulas, the *coarsest* filtration M^* of a model M is defined by putting $R^*[u][v]$ if and only if *both* of the following conditions are met:

(a) If $\Box\varphi \in \Gamma$ and $M \models_u \Box\varphi$ then $M \models_v \varphi$;

(b) If $\Diamond\varphi \in \Gamma$ and $M \models_v \varphi$ then $M \models_u \Diamond\varphi$.

97. Proposition: The coarsest filtration M^* is indeed a filtration.

Proof. Given the definition of R^*, the only condition that is left to verify is the implication from Ruv to $R^*[u][v]$. Assuming Ruv, suppose $\Box\varphi \in \Gamma$ and $M \models_u \Box\varphi$; then obviously $M \models_v \varphi$. Similarly if $\Diamond\varphi \in \Gamma$ and $M \models_v \varphi$ then $M \models_u \Diamond\varphi$. ■

98. Proposition: If Γ is finite then any filtration M^* of a model M through Γ is also finite.

Proof. If $u \equiv v$ then, by Theorem 92, the set of $\varphi \in \Gamma$ that are true at u is the same as the set of $\varphi \in \Gamma$ that are true at v. So to each $[u] \in W^*$ we can assign a *distinct* subset of Γ. Hence if Γ contains n sentences the cardinality of W^* is no greater than 2^n. ■

99. Definition: A system Σ of modal logic is said to have the *finite model property* if whenever a formula φ is true at a world in a model of Σ then φ is true at a world in a *finite* model of Σ.

100. Proposition: Let $\mathscr{U}$ be the class of universal models (see Proposition 45) and $\mathscr{U}_{\mathsf{Fin}}$ the class of all finite universal models. Then any formula φ is valid in $\mathscr{U}$ if and only if it is valid in $\mathscr{U}_{\mathsf{Fin}}$.

Proof. Finite universal models are universal models, so the left-to-right direction is trivial. For the right-to left direction, suppose that φ is false at some world w in a universal model M. Let Γ contain φ as well as all of its subformulas; clearly Γ is finite. Take a filtration M^* of M; then M^* is finite by Proposition 98, and by Theorem 92 φ is false at $[w]$ in M^*. It remains to observe that M^* is also universal: given u and v, by hypothesis Ruv and by Definition 91, part (b), also $R^*[u][v]$. ■

101. Corollary: $\mathsf{S5}$ has the finite model property.

Proof. By Propositions 45 and 100, if φ is true at a world in some reflexive and euclidean model then it is true at a world in a finite universal model (universal models are obviously reflexive and euclidean). ■

The finite model property gives us an easy way to show that systems of modal logic given by schemas are *decidable* (i.e., that there is a computable procedure to determine whether a formulas is provable in the system or not).

102. Theorem: $\mathsf{S5}$ is decidable.

Proof. Let φ be given, and suppose the propositional variables occurring in φ are among $p_1, \dots, p_k$. Since for each n there are only finitely many models with n worlds assigning a value to $p_1, \dots, p_k$, we can enumerate, *in parallel*, all the theorems of $\mathsf{S5}$ by generating proofs in some systematic way; and all the models containing $1, 2, \dots$ worlds and checking whether φ fails at a world in some such model. Eventually one of the two parallel processes will give an answer, as by Theorem 87 and Corollary 101, either φ is provable or it fails in a finite universal model. ■

The above proof works for $\mathsf{S5}$ because filtrations of universal models are automatically universal. The same holds for reflexivity and seriality, but more work is needed for other properties, as shown in the following exercises.

103. Exercise: Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

104. Exercise: Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.

$C_1(u,v)$:	if $\Box\varphi \in \Gamma$ and $M \models_u \Box\varphi$ then $M \models_v \varphi$; and if $\Diamond\varphi \in \Gamma$ and $M \models_v \varphi$ then $M \models_u \Diamond\varphi$;
$C_2(u,v)$:	if $\Box\varphi \in \Gamma$ and $M \models_v \Box\varphi$ then $M \models_u \varphi$; and if $\Diamond\varphi \in \Gamma$ and $M \models_u \varphi$ then $M \models_v \Diamond\varphi$;
$C_3(u,v)$:	if $\Box\varphi \in \Gamma$ and $M \models_u \Box\varphi$ then $M \models_v \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $M \models_v \Diamond\varphi$ then $M \models_u \Diamond\varphi$;
$C_4(u,v)$:	if $\Box\varphi \in \Gamma$ and $M \models_v \Box\varphi$ then $M \models_u \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $M \models_u \Diamond\varphi$ then $M \models_v \Diamond\varphi$;

Figure 10: Conditions on possible worlds for defining filtrations.

11. More on filtrations

105. Definition: Let Γ be closed under subformulas and $M = (W, R, V)$ a model. Then we can define conditions on pairs of worlds u, v as given in the table of Figure 10.

106. Theorem: Let $M = (W, R, P)$ be a model, Γ closed under subformulas. Let W^* and V^* be defined as in Definition 91. Then:

(a) If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \,\&\, C_2(u,v)$ then R^* is symmetric, and $M^* = (W^*, R^*, V^*)$ is a filtration if M is symmetric.

(b) If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \,\&\, C_3(u,v)$ then R^* is transitive, and $M^* = (W^*, R^*, V^*)$ is a filtration if M is transitive.

(c) If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \,\&\, C_2(u,v) \,\&\, C_3(u,v) \,\&\, C_4(u,v)$ then R^* is symmetric and transitive, and $M^* = (W^*, R^*, V^*)$ is a filtration if M is symmetric and transitive.

(d) If R^* is defined as $R^*[u][v]$ if and only if $C_1(uv) \,\&\, C_3(u,v) \,\&\, C_4(u,v)$ then R^* is transitive and euclidean, and $M^* = (W^*, R^*, V^*)$ is a filtration if M is transitive and euclidean.

Proof. We do part (a) as an example: it's immediate that R^* is symmetric, since $C_1(u,v) \iff C_2(v,u)$ and $C_2(u,v) \iff C_1(v,u)$. So it's left to show that if M is symmetric then M^* is a filtration through Γ. By condition $C_1(u,v)$ we get that: if $\Box\varphi \in \Gamma$ and $M \models_u \Box\varphi$ then $M \models_v \varphi$, and if $\Diamond\varphi \in \Gamma$ and $M \models_v \varphi$ then $M \models_u \Diamond\varphi$. So all we need is that Ruv implies $R^*[u][v]$.

So suppose Ruv, to show $R^*[u][v]$ we need $C_1(u,v) \,\&\, C_2(u,v)$. For C_1: if $\Box\varphi \in \Gamma$ and $M \models_u \Box\varphi$ then also $M \models_v \varphi$ (since Ruv); and similarly if $\Diamond\varphi \in \Gamma$ and $M \models_v \varphi$ then $M \models_u \Diamond\varphi$. For C_2: if $\Box\varphi \in \Gamma$ and $M \models_v \Box\varphi$ then Ruv implies Rvu by symmetry, so that $M \models_u \varphi$; similarly if $\Diamond\varphi \in \Gamma$ and $M \models_u \varphi$ then $M \models_v \Diamond\varphi$(since Rvu by symmetry). ■

This approach does not work in the case of models that are euclidean or serial and euclidean. Consider the model at the top of Figure 11, which is both euclidean and serial. Let $\Gamma = \{p, \Box p\}$. When taking a filtration through Γ, then $[w_1] = [w_3]$ since w_1 and w_3 are the only worlds that agree on Γ. Any filtration will also have the arrow inherited from M, as depicted in Figure 12. But we cannot add arrows to that model in order to make it euclidean, for then there would be a double arrow between

w_2 and w_4, and hence also between w_2 and w_5. But $\Box p$ is true at w_2 while p is false at w_5.

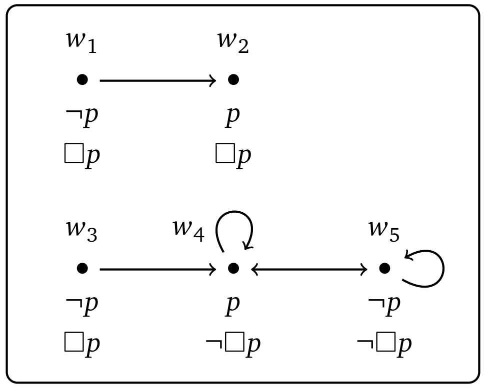


Figure 11: A serial and euclidean model

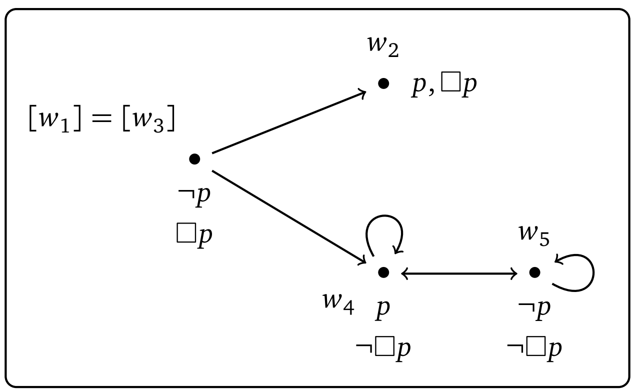


Figure 12: The filtration from the model in Figure 11.

In particular, it is not enough to consider filtrations through arbitrary Γ's closed under subsentences. Instead we need to consider sets Γ that are *modally closed* (see Definition 89). Such sets of sentences are infinite, and therefore do not lead immediately to the decidability of the corresponding system.

107 . Theorem: Let Γ be modally closed and $M = (W, R, V)$. If $M^* = (W^*, R^*, V^*)$ is a coarsest filtration of M, then M^* is symmetric, transitive or euclidean if M is symmetric, transitive, or euclidean, respectively.

Proof. The proof of transitivity uses the validity of both 4 and $4_\Diamond$ in all transitive models, and likewise euclideanness uses the fact that both 5 and $5_\Diamond$ are valid in all euclidean models, and the proof of symmetry likewise uses both B and $B_\Diamond$.

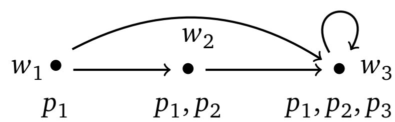
If M^* is a coarsest filtration, then by definition $R^{[}u][v]$ holds if and only if $C_1(u, v)$. For transitivity, suppose $C_1(u, v)$ and $C_1(v, w)$: to show $C_1(u, w)$ suppose $M \models_u \Box\varphi$; then $M \models_u \Box\Box\varphi$; since $\Box\Box\varphi \in \Gamma$ by closure, also by $C_1(u, v)$, $M \models_v \Box\varphi$ and by $C_1(v, w)$, also $M \models_w \varphi$. The case for $\Diamond\varphi$ is similar. ■

Part III

Problem sets

12. Problem set 1: understanding Kripke semantics

1. Problem (60 points)**:** Consider the following model M for the language comprising p_1, p_2, p_3 as the only propositional variables:



Are the following formulas and schemas true in the model M, i.e., true at every world in M? Explain.

(*a*) $p \to \Diamond p$ (for p atomic);

(*b*) $\varphi \to \Diamond\varphi$ (for φ arbitrary);

(*c*) $\Box p \to p$ (for p atomic);

(*d*) $\neg p \to \Diamond\Box p$ (for p atomic);

(*e*) $\Diamond\Box\varphi$ (for φ arbitrary);

(*f*) $\Box\Diamond p$ (for p atomic).

2. Problem (20 points)**:** For each of the following formulas find a model $M = (W, RV)$ and a world $w \in W$ such that the formula fails at w:

(*a*) $p \to \Box\Box p$;

(*b*) $\Box(p \vee q) \to (\Box p \vee \Box q)$.

3. Problem (20 points)**:** For each of the following schemas find a model **M** such that every instance of the schema is true in **M**:

(*a*) $\varphi \to \Diamond\Diamond\varphi$;

(*b*) $\Diamond\varphi \to \Box\varphi$.

13. Problem set 2: provability

1. Definition: Given a normal system Σ of modal logic, say that φ is *provable* in Σ, written $\Sigma \vdash \varphi$, if and only if there is a proof of φ from axioms in Σ using MP and RN. Equivalently, φ belongs to the smallest set of sentences containing Σ and closed under tautological implication and RK.

2. Problem (24 points)**:** Provide K-proofs of the following:

(*a*) $\Diamond\neg\bot \to (\Box\varphi \to \Diamond\varphi)$;

(*b*) $\Box(\varphi \vee \psi) \to (\Diamond\varphi \vee \Box\psi)$;

(*c*) $(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$.

For ease of reference, we restate here Theorem 60:

Soundness Theorem: if schemas $\mathsf{S}_1, \ldots, \mathsf{S}_n$ are valid in the classes of models $\mathscr{C}_n, \ldots, \mathscr{C}_n$, respectively, then $\mathsf{KS}_1 \ldots \mathsf{S}_n \vdash \varphi$ implies that φ is valid in the class of models $\mathscr{C}_n \cap \ldots \cap \mathscr{C}_n$.

3. Definition: An inference rule of the form

$$\frac{\varphi_1 \cdots \varphi_n}{\psi} \qquad (*)$$

is *admissible* in a system Σ if and only if, whenever $\Sigma \vdash \varphi_i$ for $i = 1, \ldots, n$, then also $\Sigma \vdash \psi$. In other words, adding the new rule to the system does not change the set of provable formulas.

4. Definition: An inference rule of the form $(*)$ is *derivable* in a system Σ if and only if $\Sigma \vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots)$.

5. Problem (6 points)**:** Show that if a rule is derivable in Σ then it is admissible in Σ.

6. Definition: Define a function σ from formulas into formulas recursively by setting:

$$\begin{aligned}
\sigma(p) &= p; \\
\sigma(\bot) &= \bot; \\
\sigma(\neg\varphi) &= \neg\sigma(\varphi); \\
\sigma(\varphi \to \psi) &= \sigma(\varphi) \to \sigma(\psi); \\
\sigma(\Box\varphi) &= \varphi.
\end{aligned}$$

So σ erases the outermost occurrence of $\Box$. For instance, compute $\sigma(\Box(\Diamond A \to \Diamond\Box B))$ and $\sigma(\Diamond A \to \Diamond\Box B)$.

7. Problem (20 points)**:** Show that if $\mathsf{K} \vdash \varphi$ then $\mathsf{K} \vdash \sigma(\varphi)$ (use induction on the number of lines in the proof).

8. Problem (10 points)**:** Show that the following inference rules are admissible in K:

$$(a)\ \frac{\Diamond\varphi}{\varphi} \qquad (b)\ \frac{\Box\varphi \to \Box\psi}{\varphi \to \psi} \qquad (c)\ \frac{\Box\varphi}{\varphi}.$$

If Σ and Σ' are two systems such that $\Sigma \subseteq \Sigma'$ then (obviously) every theorem of Σ is a theorem of Σ'. The situation is different, however, for rules. If a rule of the form $(*)$ is admissible in Σ, it does *not* follow that it is admissible in Σ' as well: the latter has *more theorems* than the former, and so the harder it is to show that a given rule is admissible.

9. Problem (10 points)**:** Show that $\mathsf{KB} \vdash \Box(\varphi \to \Diamond\Diamond\varphi)$, but $\mathsf{KB} \nvdash \varphi \to \Diamond\Diamond\varphi$. *Cheat*: for half the credit, use completeness of KB with respect to symmetric models to show that $\mathsf{KB} \vdash \Box(\varphi \to \Diamond\Diamond\varphi)$.

10. Problem (10 points)**:** Show that $\mathsf{K5} \vdash \Box(\varphi \to \Diamond\varphi)$, but $\mathsf{K5} \nvdash \varphi \to \Diamond\varphi$. *Cheat*: for half the credit, use completeness of $\mathsf{K5}$ with respect to euclidean models to show that $\mathsf{K5} \vdash \Box(\varphi \to \Diamond\varphi)$.

11. Problem (10 points)**:** Is the rule (c) above admissible in KB? in $\mathsf{K5}$?

12. Problem (10 points)**:** We know from Problem 1 that all derivable rules are admissible. Show that the converse is not true.

14. Problem set 3: p-morphisms

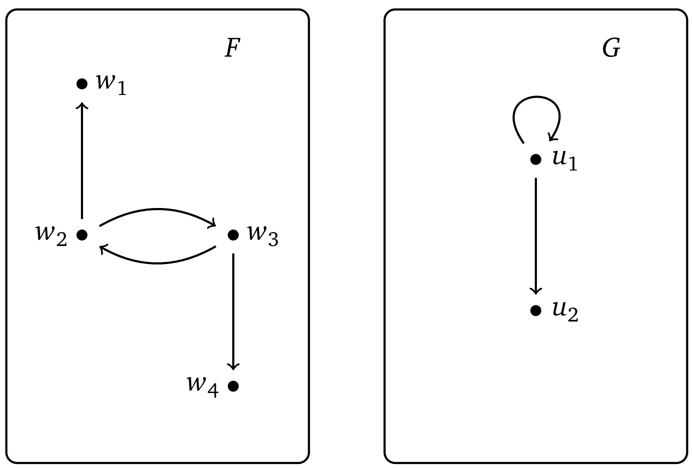
1. Definition: A frame is *irreflexive* if for any world w in the frame it's *not* the case that $R(w,w)$, i.e., no world is accessible from itself.

2. Definition: Given two frames $F=(W,R)$ and $G=(X,S)$, a p-morphism of the first onto the second is a function $\pi : W \to X$ such that:

- π is surjective;
- if $R(u,v)$ then $S(\pi(u),\pi(v))$;
- if $S(\pi(u),w)$ then there is a $v \in W$ such that $\pi(v)=w$ and $R(u,v)$.

Given two models $M=(W,R,U)$ and $N=(X,S,V)$, a p-morphism of the first onto the second is a p-morphism π of the corresponding frames satisfying the additional condition that for all $w \in W$ and propositional variable p: $w \in U(p)$ if and only if $\pi(w) \in V(p)$.

3. Problem (10 points)**:** Show that there is a p-morphism between the following two frames F and G, where $W=\{w_1,w_2,w_3,w_4\}$ and $X=\{u_1,u_2\}$ and R and S are as depicted:



4. Problem (15 points)**:** Let the frames F and G be like in Problem 3, and assume that the language only comprises propositional variable p_1,p_2,p_3 for simplicity.

(*a*) find valuations $U : \{p_1,p_2,p_3\} \to \mathscr{P}(W)$ and $V : \{p_1,p_2,p_3\} \to \mathscr{P}(X)$ such that there is a p-morphism between the models $M=(W,R,U)$ and $N=(X,S,V)$.

(*b*) Are there valuations U and V such that the corresponding models are not p-morphic?

5. Problem (15 points)**:** Let π be a p-morphism between models $M=(W,R,U)$ and $N=(X,S,V)$. Show that for any formula φ and world $w \in W$,

$$M \models_w \varphi \text{ if and only if } N \models_{\pi(w)} \varphi.$$

6. Problem (15 points)**:** Show that for any model N on G there is a model M on F and a p-morphism π from M onto N.

7. Problem (10 points)**:** Show that the converse to Problem 6 does not hold, i.e., that there is a model on F that is not p-morphic to any model on G.

8. Problem (10 points)**:** Show that if a formula φ is valid on F then it is valid on G.[1]

9. Problem (10 points)**:** Show that the converse of Problem 8 does not hold, i.e., that there is a formula φ that is valid on G but not on F.[2]

10. Problem (15 points)**:** The schema T, i.e., $\Box\varphi \to \varphi$ is valid in a frame if and only if the frame is reflexive. Is there a schema that is valid in a frame if and only if the frame is *irreflexive*?

15. Problem set 4: pebble games over Kripke frames

1. Definition (Two-pebble games on Kripke structures)**:** Given two Kripke models $M = (W, R, U)$ and $N = (X, S, V)$ and worlds $u \in W$ and $v \in X$, the *two-pebble game* $G(M, u : N, v)$ between Abelard ("Abe the Spoiler") and Eloïse ("Elly the Defender") is defined as follows.

Abelard and Eloïse share two pebbles, which at the beginning of the game are placed on the worlds u and v. A *round* in the game begins with Abelard's sliding one of the pebbles along the accessibility relation (R in M or S in N). Eloïse likewise responds by sliding the other pebble along S or R, respectively. In this way, two worlds u' and v' are identified such that Ruu' and Svv'. If neither player wins the game at this round (as explained below), then both players advance and play a round of the game $G(M, u' : N, v')$. Suppose the pebbles are on worlds $u \in W$ and $v \in X$; this configuration is assessed as a *win* in the game $G(M, u : N, v)$ for one or the other players based on whether the following conditions hold:

(a) If there is an atomic sentence p true at u and false at v, or vice-versa, it's an automatic win for Abelard.

(b) If the worlds u and v are both terminal, it's an automatic win for Eloïse.

(c) If at any point Eloïse cannot slide her pebble to match Abelard's move, he wins.

(d) If the game goes on forever, that counts as a win for Eloïse.

2. Definition: A *strategy* for either player in the game $G(M, u : N, v)$ is just a function prescribing a response to each possible move of the other player. A *winning strategy* is strategy that guarantees a win for the player following it.

3. Problem: Show that if Abelard does *not* have a winning strategy in the game $G(M, u : N, v)$ then for any modal formula φ we have $M \models_u \varphi$ if and only if $N \models_v \varphi$.

[1] Warning. This requires showing that if φ is true in any model on F then it is true in any model on G. Use Problem 6.

[2] Hint. Consider a formula that says: 'At any non-terminal world: if φ is true and also true at any accessible terminal world, then φ is necessary.'