Basic Derivation Rules for Modal Logic

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The first derivation system for Modal Sentential Logic in the Fitch style was given by Fitch himself. The central idea was to allow a kind of construction in a derivation, the *restricted sub-derivation*, which is not a feature of his derivation system for non-modal sentential logic. The restriction is on the use of Reiteration. Only certain sentences (or their components) are allowed to be reiterated into a restricted sub-derivation. This restrictive form of Reiteration is embodied in new modal reiteration rules. In this module, we will focus on several versions of such a rule. We will also look at some rules for introducing and eliminating modal operators—rules which work in tandem with the restricted reiteration rules. The goal of the derivation systems employing these rules is to mirror the semantical results which were the subject of the last module.

1 Derivation Rules for the '$\Box$' Operator

We shall begin our treatment of the derivation rules with the '$\Box$' operator. There are two relevant semantical rules governing its use (the first stated only partially here):

SR-$\Box$ If $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=$\mathbf{T}$ at all worlds $\mathbf{w}_i$ in $\mathbf{I}$ such that $\mathbf{Rww}_i$, then $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=$\mathbf{T}$.

SR-$\Box$C If $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=$\mathbf{T}$, then $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=$\mathbf{T}$ at all worlds $\mathbf{w}_i$ in $\mathbf{I}$ such that $\mathbf{Rww}_i$.

When we use the rule **SR-$\Box$**C to reason about what follows from the truth of $\Box\alpha$ at a world $\mathbf{w}$, we assume that $\mathbf{w}_i$ is an arbitrary world accessible to $\mathbf{w}$ and note that α is assigned true there. This is the role played by the restricted scope line: it represents an arbitrary accessible possible world. A vertical line flanked on its first line by a '$\Box$' on its left graphically represents the relation of accessibility from a given world. We take the (implicitly assumed) truth of sentences outside the restricted scope line to be truth at a world, and truth inside the scope line to be truth at a world accessible to that world.

Restricted Scope Line

$$\begin{array}{r|l} \Box & \\ & \\ & \\ & \end{array}$$

Where $\Box\alpha$ occurs in a derivation, we can import α across a restricted scope line which itself is to the immediate right of the scope line in which $\Box\alpha$ occurs.[1] This rule is known as 'Strict Reiteration for '$\Box$'' or 'SR-$\Box$.'[2] (This rule should not be confused with the semantical rule **SR-**$\Box$, whose name uses bold-face letters.)

Strict Reiteration for '$\Box$'

$$\begin{array}{|r|ll}
 & \Box\alpha & \text{Already Derived} \\
\Box & \;. & \\
 & \;. & \\
 & \;. & \\
 & \alpha & \text{SR-}\Box
\end{array}$$

The rules for the truth-functional operators work in the usual way within a restricted scope line. This reflects the fact that the semantical rules for those operators function in the normal way at any possible world.

It is crucial that importation of earlier steps across restricted scope lines be limited to what is permitted by the rule of Strict Reiteration. What is outside the restricted scope line reflects a truth-value assignment at a world, and that assignment may be different from the one represented inside the restricted scope line. Suppose a sentence α has the value **T** at a world **w**, and that $\mathbf{Rww_1}$. We cannot in general expect that the value at **w** will be preserved at $\mathbf{w}_1$. If α is a sentence letter such as 'P', it may well be true at **w** but false at $\mathbf{w}_1$. Since each step in a derivation is supposed to represent the assignment of truth to a sentence, we might get the truth-value of '*P*' wrong by writing '*P*' to the right of a restricted scope line when

[1] Note that only one restricted scope line may be crossed. The semantical rule allows only truth at all worlds accessible to the given world.

[2] The rule might more intuitively be thought of as a $\Box$ Elimination rule, but in keeping with common practice, we will reserve that denomination for another rule to be introduced in a subsequent module.

it occurs to the left. So we shall stipulate that Reiteration may not be applied across a restricted scope line, though any rule may be applied when the application takes place wholly within a restricted scope line.

Reiteration for Modal Derivations

α	Already Derived
$\ldots$	
α	R
$\quad\beta$	Assumption
$\quad\alpha$	R

Provided that any scope line crossed is not restricted.

By itself, the use of a restricted scope line is of no value, just as it is of no value semantically merely to know that a sentence is true at an accessible possible world. What we would like to be able to do is to use the information about the truth-value at the accessible world to discover the value of a sentence at the "home" world—the world to which the accessible world is accessible.

Here is an example of the beginning of a derivation using **SR-□**.

Illustration of the use of SR-□

1	$\Box(P \wedge Q)$	Assumption
2	$\Box\quad P \wedge Q$	1 SR-□
3	$\quad P$	2 ∧ E

Given the intended interpretation of the strict scope line, what we are representing is the truth of '*P*' at an arbitrary accessible world. If '*P*' is true at an arbitrary accessible world, it is true at all accessible worlds. This means that '□*P*' should be true at the home world, as SR-□ requires.

Illustration of the need for a further rule

1	$\Box(P \wedge Q)$	Assumption
2	$\Box\quad P \wedge Q$	1 SR-□
3	$\quad P$	2 ∧ E
4	$\Box P$	Rule Needed Here

We shall generalize this reasoning and state a rule which allows us to end a strict scope line and prefix a '□' to the last step not in the scope of any assumption within the strict scope line. This is the rule '□ Introduction,' or '□ I.'[3] It holds for all *MSL* derivation systems which conform to the semantics given in the last section.[4]

□ Introduction

$$\begin{array}{l|l} \Box & . \\ & . \\ & . \\ & \alpha \end{array}$$
$$\Box\alpha \qquad \Box\ \mathrm{I}$$

Provided that α is not in the scope of any assumption within the strict scope line.

In annotating the use of the rule, we will indicate the whole series of steps within the strict scope line.

Strict Reiteration is not required for the use of □ introduction, since we may establish results entirely within the strict scope line. For example, we can derive '□~(P ∧ ~P)' using some rules of *SD* and the rule of □ Introduction.

To Prove: $\vdash \Box{\sim}(P \land {\sim}P)$

$$\begin{array}{r|l|l|ll} 1 & \Box & & \underline{P \land {\sim}P} & \text{Assumption} \\ 2 & & & P & 1 \land \mathrm{E} \\ 3 & & & {\sim}P & 1 \land \mathrm{E} \\ 4 & & \multicolumn{2}{l}{{\sim}(P \land {\sim}P)} & \text{1-3} \sim \mathrm{I} \\ 5 & \multicolumn{3}{l}{\Box{\sim}(P \land {\sim}P)} & \text{1-4}\ \Box\ \mathrm{I} \end{array}$$

To show more precisely how the reasoning in a modal derivation parallels semantical reasoning, we shall juxtapose a derivation to prove $\{\Box P, \Box(P \supset Q)\} \vdash \Box Q$ with a specially-formulated semantical proof that $\{\Box P, \Box(P \supset Q)\} \vDash \Box Q$.

[3]We will not here give a □ Elimination rule. One such rule would allow the removal of the '□' operator from $\Box\alpha$, so that α can be written down. But nothing about the semantical rule for the '□' requires that if $\mathbf{v_I}$ ($\Box\alpha$, $\mathbf{w}$)=$\mathbf{T}$, then $\mathbf{v_I}$(α, $\mathbf{w}$)=$\mathbf{T}$. Such a rule will be forthcoming when we consider semantical systems strong enough to support it.

[4]These systems are called "normal" systems of Modal Sentential Logic. The way in which derivational systems conform to semantical systems will be explained in the context of system *K*, which we will examine in the next module.

To prove: $\{\Box P, \Box(P \supset Q)\} \vdash \Box Q$

1	$\Box P$	Assumption
2	$\Box(P \supset Q)$	Assumption
3	$\Box$ P	1 SR-$\Box$
4	$P \supset Q$	2 SR-$\Box$
5	Q	3 4 $\supset$ E
6	$\Box Q$	3-5 $\Box$ I

We cast the semantical proof in a Fitch-style derivation *in the meta-language*. To avoid confusion with object-language derivations, we shall use special meta-logical quantifiers and operators. The universal quantifier symbol is '$\prod$' and the existential quantifier symbol is '$\sum$.' The negation sign is '$\neg$,' the conjunction sign is '$\bigwedge$,' the disjunction sign is '$\bigvee$,' the conditional sign is '$\rightarrow$,' and the biconditional sign is '$\leftrightarrow$.' The meta-logical derivations to follow will not be rigorous, in that some obvious steps will be skipped because they obscure the structural similarities to be illustrated.

Skeleton of a semantical proof of $\{\Box P, \Box(P \supset Q)\} \vDash \Box Q$

1	$\mathbf{v_I}(\Box P, \mathbf{w}) = \mathbf{T}$	Assumption
2	$\mathbf{v_I}(\Box(P \supset Q), \mathbf{w}) = \mathbf{T}$	Assumption
3	$\mathbf{Rww}_i$	Assumption
4	$\mathbf{v}(P, \mathbf{w}_i) = \mathbf{T}$	1 **SR-**$\Box$C
5	$\mathbf{v}(P \supset Q, \mathbf{w}_i) = \mathbf{T}$	2 **SR-**$\Box$C
6	$\mathbf{v}(Q, \mathbf{w}_i) = \mathbf{T}$	4 5**SR-**$\supset$
7	$\mathbf{Rww}_i \rightarrow \mathbf{v}(Q, \mathbf{w}_i) = \mathbf{T}$	3-6 $\rightarrow$ I
8	$\mathbf{v}(\Box Q, \mathbf{w}) = \mathbf{T}$	7 **SR-**$\Box$

The use of quantifiers is suppressed here as is the statement of the semantical rules. Due to the close resemblance in this case between the structure of the meta-logical proof of semantical entailment and the derivation, we can think of the latter as an abbreviation of the former. The reader can verify that other derivations can be recast in a similar way, so that the derivation system as a whole can be thought of as providing an abbreviated version of semantical proofs.

2 Derivation Rules for '$\Diamond$' Operator

We will consider two distinct sets of derivation rules for the '$\Diamond$' operator. The first set consists of two derived rules based on the rules for the '$\Box$' operator. The second set consists of rules specifically designed for the '$\Diamond$' operator. The first set has the advantage of producing derivations that are easily converted into derivations in the system for the '$\Box$' operator. Using the first set will allow the derivation of whatever can be derived in the '$\Box$'-based system. The second set appears to be incomplete relative to the '$\Box$'-based system, in the sense that it seems that it will not allow all the derivations possible in it.

2.1 Impossibility Rules

One way to generate a set of rules for the '$\Diamond$' operator is to proceed in the manner of Lewis and base the rules on the combination '$\sim\Diamond$,' which is intended to represent impossibility. According to the semantics for the '$\Box$' and the '$\Diamond$,' we get the following semantical equivalence.

$\Box{\sim}\alpha$ is equivalent to ${\sim}\Diamond\alpha$.

Proof. $\mathbf{v_I}(\Box{\sim}\alpha, \mathbf{w})$=$\mathbf{T}$ iff for all $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, $\mathbf{v_I}({\sim}\alpha, \mathbf{w}_i)$=$\mathbf{T}$ iff for all $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=$\mathbf{F}$ iff $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$ = $\mathbf{F}$ iff $\mathbf{v_I}({\sim}\Diamond\alpha, \mathbf{w})$=$\mathbf{T}$.

With this equivalence in hand, we can re-write the rules of Strict Reiteration and $\Box$ Introduction. First, we present Strict Reiteration, with '$\Box\sim$.'

Strict Reiteration for a Negated Sentence

	$\Box{\sim}\alpha$	Already Derived
$\Box$	.	
	.	
	.	
	${\sim}\alpha$	SR-$\Box$

Then we substitute the semantically equivalent '${\sim}\Diamond\alpha$' for '$\Box{\sim}\alpha$.'

Strict Reiteration for $\sim\Diamond$

$$
\begin{array}{l|l}
\sim\Diamond\alpha & \text{Already Derived} \\
\square\ \vert\ . & \\
\quad\ \vert\ . & \\
\quad\ \vert\ . & \\
\quad\ \vert\ \sim\alpha &
\end{array}
$$

The soundness of the rule can be seen from this semantical argument. If $\mathbf{v_I}(\sim\Diamond\alpha,\mathbf{w}) = \mathbf{T}$, then $\mathbf{v_I}(\Diamond\alpha,\mathbf{w})=\mathbf{F}$, so there is no accessible world $\mathbf{w}_i$ at which $\mathbf{v_I}(\alpha,\mathbf{w}_i)=\mathbf{T}$. So given an accessible world $\mathbf{w}_i$, we must assign α the value **F** there, in which case the value of $\sim\alpha$ must be **T** at $\mathbf{w}_i$, which was to be shown.

The same kind of reasoning can be used to produce a derived rule for the introduction of '$\sim\Diamond$.'

$\square$ Introduction for a Negated Sentence

$$
\begin{array}{l|l}
\square\ \vert\ . & \\
\quad\ \vert\ . & \\
\quad\ \vert\ . & \\
\quad\ \vert\ \sim\alpha & \\
\square\sim\alpha & \square\ \mathrm{I}
\end{array}
$$

Given the semantic equivalence, we get:

$\sim\Diamond$ Introduction

$$
\begin{array}{l|l}
\square\ \vert\ . & \\
\quad\ \vert\ . & \\
\quad\ \vert\ . & \\
\quad\ \vert\ \sim\alpha & \\
\sim\Diamond\alpha & \sim\Diamond\ \mathrm{I}
\end{array}
$$

Provided that $\sim\alpha$ is not in the scope of any assumption within the strict scope line.

This rule is also sound, given the basic semantics. Suppose an aribitrary world $\mathbf{w}_i$ is accessible to a world **w**. If a sentence ${\sim}\alpha$ is assigned **T** at $\mathbf{w}_i$, then α is assigned **F** at that world. Since $\mathbf{w}_i$ is arbitrary, α is assigned **F** at all worlds accessible to **w**, in which case $\Diamond\alpha$ is assigned **F** at **w**. Thus ${\sim}\Diamond\alpha$ is assigned **T** at **w**, which was to be shown.

With these two rules we can derive the semantical equivalents of any necessity-sentence that can be derived using the necessity rules. For example, we have seen that from '$\Box$P' and '$\Box$(P ⊃ Q)' we can derive '$\Box$Q'. This gives the same semantical results as deriving '${\sim}\Diamond$P' from '${\sim}\Diamond$Q' and '${\sim}\Diamond$(P ∧ ${\sim}$Q)'. (The second premise can also be read as 'P ⥽ Q.')

Exercise. Explain why these two derivations give the same semantic results.

To Prove: $\{{\sim}\Diamond Q, {\sim}\Diamond(P \wedge {\sim}Q)\} \vdash {\sim}\Diamond P$

1	${\sim}\Diamond Q$	Assumption
2	${\sim}\Diamond(P \wedge {\sim}Q)$	Assumption
3	$\Box$ ${\sim}Q$	1 SR-${\sim}\Diamond$
4	${\sim}(P \wedge {\sim}Q)$	2 SR-${\sim}\Diamond$
5	P	Assumption
6	${\sim}Q$	3 Reiteration
7	$P \wedge {\sim}Q$	5 6 ∧ I
8	${\sim}(P \wedge {\sim}Q)$	4 Reiteration
9	${\sim}P$	5-8 ${\sim}$ 1
10	${\sim}\Diamond P$	3-9 ${\sim}\Diamond$ I

Another derivation shows the ${\sim}\Diamond$ Introduction rule working by itself. We can derive '${\sim}\Diamond$(P ∧ ${\sim}$P)' from no undischarged assumptions, showing that it is a theorem of given the basic derivation system of Modal Sentential Logic.

To Prove: $\vdash \sim\Diamond(P \wedge \sim P)$

1	$\Box$ $\mid\ \mid\ P \wedge \sim P$	Assumption
2	$\mid\ \mid\ P$	$1 \wedge$ E
3	$\mid\ \mid\ \sim P$	$1 \wedge$ E
4	$\mid\ \sim(P \wedge \sim P)$	1-3 $\sim$ I
5	$\sim\Diamond(P \wedge \sim P)$	1-4 $\sim\Diamond$ I

2.2 Possibility Rules

Despite their completeness relative to the necessity rules, the two impossibility rules seem less than satisfactory because they involve a non-modal operator. They are not pure possibility rules. We shall give a derived pure possibility rule shortly. This rule was given by Fitch in the original adaptation of derivability rules to modal logic.[5]

As with the '□' operator, we take our cue from the semantical rules, the first of which is stated partially.

SR-$\Diamond$ If $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** at some world $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$, then $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=**T**.

SR-$\Diamond$C If $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=**T**, then $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** at some world $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$.

The rules for '$\Diamond$' work somewhat differently from the Strict Reiteration and the Introduction rules we have provided. The clue for their structure is found in the fact that **SR-**$\Diamond$C involves an existential rather than a universal quantifier in the meta-language. When we reason from the truth of $\Diamond\alpha$ at a world **w**, we can only infer that there is at least one accessible world $\mathbf{w}_i$ at which α is true. Because a strict scope line is supposed to represent an arbitrary world, it may not represent a world at which α is true. The only thing we can do is to *assume* that α is true at an arbitrary world and consider what would happen *if* this were so.

To represent this kind of reasoning, we will use a new kind of strict scope line, which contains a *strict assumption*. It is written like an assumption line, but with a '□' to the left of it. Then the reiterated sentence α is written as an assumption.

[5] It is the opinion of the author that no combination of pure possibility rules will be complete relative to the basic modal semantics. In particular, the theorem just demonstrated seems not to be derivable without impossibility rules of some kind.

Strict Reiteration for '$\Diamond$'

$$
\begin{array}{l|l|ll}
 & \multicolumn{2}{l}{\Diamond\alpha} & \text{Already Derived} \\
 & \Box & \alpha & \text{SR-}\Diamond \\
\cline{3-3}
 & & . & \\
 & & . & \\
 & & . & \\
\end{array}
$$

As with the necessity operator, we would like to be able to use the information within the strict scope line to establish something outside it. When we have the situation where α is assumed to be true at some world $\mathbf{w}_i$ accessible to **w**, and some sentence β is found to be true there as well, we can say by **SR-**$\Diamond$ that $\Diamond\beta$ is true at **w**. To reflect this condition, we lay down a rule of $\Diamond$ Elimination.[6] When the strict assumption is discharged, the last step β (not within any other scope lines) is brought out as $\Diamond\beta$.

$\Diamond$ Elimination

$$
\begin{array}{l|l|ll}
 & \multicolumn{2}{l}{\Diamond\alpha} & \text{Already Derived} \\
 & \Box & \alpha & \text{SR-}\Diamond \\
\cline{3-3}
 & & . & \\
 & & . & \\
 & & . & \\
 & & \beta & \\
 & \multicolumn{2}{l}{\Diamond\beta} & \Diamond\ \text{E} \\
\end{array}
$$

Provided that β is not in the scope of any assumption within the strict scope line.

Note that an application of SR-$\Diamond$ always is accompanied by an application of $\Diamond$ Elimination, and *vice-versa*. The only way to discharge a strict assumption is to use $\Diamond$ Elimination. And the only way to use $\Diamond$ Elimination is on the basis of a strict assumption.

[6]Calling this an "elimination" rule sounds odd, but it will turn out that we must reserve the Introduction rule for another purpose. You can think of the rule as eliminating the operator in the process of Strict Reiteration to an assumption. This rule is parallel to the Predicate Logic rule of $\exists$ Elimination.

The rule is sound given the basic semantics. Suppose that at a world **w**, $\Diamond\alpha$ is assigned the value **T**. Then by **SR**$\Diamond$**-C**, there is an accessible world $\mathbf{w}_i$ where α is true. If it is the case that β is also true at $\mathbf{w}_i$, then there is a world accessible to **w** at which β is true, so by **SR-**$\Diamond$, $\Diamond\beta$ is true at **w**, which is to be proved.

It is always open for one to make a regular assumption, rather than an SR-$\Diamond$ assumption inside a strict scope line. We did so in proving that $\vdash \Box{\sim}(P \wedge {\sim}P)$. But we could not move from step 4 to step 5 as follows:

Incorrect Derivation

1	$\Box$ $P \wedge {\sim}P$	Assumption
2	P	1 $\wedge$ E
3	${\sim}P$	1 $\wedge$ E
4	${\sim}(P \wedge {\sim}P)$	1-3 ~ I
5	$\Diamond{\sim}(P \wedge {\sim}P)$	1-4 $\Diamond$ E **ERROR**

Even if a theorem is derived inside a restricted scope line, it could not be brought out unless SR-$\Diamond$ is used as well. This is because in the basic semantics some frames will contain worlds to which there is no world accessible. We do not want to be able to derive even the possibility of α, where α is a theorem, unless we are working with the supposition that there are accessible worlds. The presence of the possibility sentence $\Diamond\alpha$ taken to be true indicates that there is an accessible world at which α is true.

To illustrate the use of the possibility rules, we will derive '$\Diamond Q$' from '$\Diamond P$' and '$\Box(P \supset Q)$'.

To Prove: $\{\Diamond P, \Box(P \supset Q)\} \vdash \Diamond Q$

1	$\Diamond P$	Assumption
2	$\Box(P \supset Q)$	Assumption
3	$\Box$ P	1 SR-$\Diamond$
4	$P \supset Q$	2 SR-$\Box$
5	Q	3 4 $\supset$ E
6	$\Diamond Q$	1 3-5 $\Diamond$ E

Note that in annotating this rule, we make reference to the possibility-sentence that was strictly reiterated.

The derivation just given can be put in semantical terms (again suppressing obvious steps), which illustrates why it is sound. We will use the meta-logical expression '$\mathbf{w}_1$' as a constant term for purposes of instantiation.

Skeleton of a semantical proof of: $\{\Diamond P, \Box(P \supset Q)\} \vDash \Diamond Q$

1	$\mathbf{v}(\Diamond P, \mathbf{w}) = \mathbf{T}$	Assumption
2	$\mathbf{v}(\Box(P \supset Q), \mathbf{w}) = \mathbf{T}$	Assumption
3	$(\sum \mathbf{w}_i)(\mathbf{Rww}_i \wedge \mathbf{v}(P, \mathbf{w}_i) = \mathbf{T})$	1 **SR-**$\Diamond$
4	$\mathbf{Rww}_1 \wedge \mathbf{v}(P, \mathbf{w}_1) = \mathbf{T}$	Assumption
5	$\mathbf{Rww}_1$	4 $\wedge$ E
6	$\mathbf{v}(P, \mathbf{w}_1) = \mathbf{T}$	4 $\wedge$ E
7	$\mathbf{v}(P \supset Q, \mathbf{w}_1) = \mathbf{T}$	2 **SR-**$\Box$
8	$\mathbf{v}(Q, \mathbf{w}_1) = \mathbf{T}$	6 7 **SR-**$\supset$
9	$\mathbf{Rww}_1 \wedge \mathbf{v}(Q, \mathbf{w}_1) = \mathbf{T}$	5 8 $\wedge$ I
10	$(\sum \mathbf{w}_i)(\mathbf{Rww}_i \wedge \mathbf{v}(Q, \mathbf{w}_i) = \mathbf{T})$	9 $\sum$ I
11	$(\sum \mathbf{w}_i)(\mathbf{Rww}_i \wedge \mathbf{v}(Q, \mathbf{w}_i) = \mathbf{T})$	3 4-10 $\sum$ E
12	$\mathbf{v}(\Diamond Q, \mathbf{w}) = \mathbf{T}$	11 **SR-**$\Diamond$

The tandem-rules SR-$\Diamond$ and $\Diamond$ Elimination can be derived from the two rules for impossibility stated in the last sub-section. That is, if we assume that we have derived $\Diamond\alpha$ and that we can derive β from α within a restricted scope line with 'α' as an assumption, we can derive $\Diamond\beta$.

Exercise: Show how the derivation works.

2.3 Strict Implication Rules

As with the one-place operators, we will motivate the derivation rules for the two-place '$\dashv\!\!\! 3$' operator on the basis of its semantical rules. Part of the rule **SR-**$\dashv\!\!\! 3$ is as follows:

SR-$\dashv\!\!\! 3$ If either $\mathbf{v_I}(\alpha,\ \mathbf{w}_i)$=**F** or $\mathbf{v_I}(\beta,\ \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$, then $\mathbf{v_I}(\alpha \dashv\!\!\! 3\ \beta,\ \mathbf{w})$=**T**.

It follows from this rule and **SR-**$\supset$ that:

SR-⥽′ If $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$, then $\mathbf{v_I}(\alpha ⥽ \beta, \mathbf{w})$=**T**.

The converse holds as well.

SR-⥽′C If $\mathbf{v_I}(\alpha ⥽ \beta, \mathbf{w})$=**T**, then $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$.

A further useful semantical rule is a variant of **SR-**⥽ which treats the condition for the truth of $\Box\alpha$ at a world as a (material) conditional.

SR-⥽″ If $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** only if $\mathbf{v_I}(\beta, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$, then $\mathbf{v_I}(\alpha ⥽ \beta, \mathbf{w})$=**T**.

Exercise: Prove the last two statements.

Given **SR-**⥽′C, we can state a rule for Strict Reiteration. If a strict conditional occurs on a line, we may open up a new strict scope line and "reiterate" the corresponding material conditional.

Strict Reiteration for '⥽'

| $\alpha ⥽ \beta$	Already Derived
| $\Box$ | .	
| | .	
| | .	
| | $\alpha \supset \beta$	SR-⥽

The rule for ⥽ Introduction follows **SR-**⥽″ and resembles the rule for $\Diamond$ Elimination. If a strict assumption of α is made and β is derived in the scope of that assumption (and not in the scope of any other assumption), then the strict scope line may be terminated, and $\alpha ⥽ \beta$ written.

$\strictif$ Introduction

$$
\begin{array}{l|l}
\Box & \alpha \\ \hline
 & . \\
 & . \\
 & . \\
 & \beta \\
\end{array}
$$

$$\alpha \strictif \beta \qquad \strictif \mathrm{I}$$

Provided that β is not in the scope of any assumption within the strict scope line.

The rule is sound given the basic semantics. Suppose an aribitrary world $\mathbf{w}_i$ is accessible to a world **w**. If a sentence α is assumed to have the value **T** at $\mathbf{w}_i$, and it can be shown that β is assigned **T** as well, then by **SR-**$\strictif''$, it follows that $\alpha \strictif \beta$ is true at **w**, which was to be shown.

It is easy to see that the following holds:

If $\{\alpha\} \vdash_{SD} \beta$, then $\vdash \alpha \strictif \beta$.

If the antecedent holds, then there is a derivation of β using only *SD* rules and with α as a strict assumption. By the rule $\strictif$ Introduction, one may then write down $\alpha \strictif \beta$ outside the scope of any assumption, so that it is a theorem. Here is an example.

To prove: $\vdash (P \wedge Q) \strictif P$

1	$\Box$ | $P \wedge Q$	Assumption
2	| P	$1 \wedge$ E
3	$(P \wedge Q) \strictif P$	1-2 $\strictif$ I

Another example of the use of the $\strictif$ rules is the derivation of one of the so-called "paradoxes of strict implication." One of the "paradoxes of material implication" is the fact that $\vdash_{SD} \alpha \supset (\beta \supset \alpha)$. On Russell's intended interpretation, this means that a true sentence α is "materially implied" by any sentence β. In Lewis's systems, it turns out that a necessarily true sentence α is "strictly implied" by any sentence β, i.e., $\vdash \Box\alpha \strictif (\beta \strictif \alpha)$.

To prove: $\vdash \Box P \strictif (Q \strictif P)$

1	$\Box$ $\Box P$	Assumption
2	$\quad$ $\Box$ Q	Assumption
3	$\quad\quad P$	1 SR-$\Box$
4	$\quad Q \strictif P$	2-3 $\strictif$ I
5	$\Box P \strictif (Q \strictif P)$	1-4 $\strictif$ I

Exercise: Prove the other "paradox of strict implication," that a necessarily false sentence strictly implies any sentence, i.e., $\vdash \Box{\sim}\alpha \strictif (\alpha \strictif \beta)$.

A parallel semantical proof illustrates the motivation for the rules we have chosen.

Skeleton of a semantical proof of: $\vDash \Box P \strictif (Q \strictif P)$

1	$\mathbf{Rww}_i$	Assumption
2	$\quad \mathbf{v}(\Box P, \mathbf{w}_i) = \mathbf{T}$	Assumption
3	$\quad\quad \mathbf{Rw}_i\mathbf{w}_j$	Assumption
4	$\quad\quad\quad \mathbf{v}(Q, \mathbf{w}_j) = \mathbf{T}$	Assumption
5	$\quad\quad\quad \mathbf{v}(P, \mathbf{w}_j) = \mathbf{T}$	2 **SR**-$\Box$C
6	$\quad\quad \mathbf{v}(Q, \mathbf{w}_j) = \mathbf{T} \rightarrow \mathbf{v}(P, \mathbf{w}_j) = \mathbf{T}$	4-5 $\rightarrow$ I
7	$\quad \mathbf{Rw}_i\mathbf{w}_j \rightarrow (\mathbf{v}(Q, \mathbf{w}_j) = \mathbf{T} \rightarrow \mathbf{v}(P, \mathbf{w}_j) = \mathbf{T})$	3-6 $\rightarrow$ I
8	$\quad \mathbf{v}(Q \strictif P), \mathbf{w}_i = \mathbf{T}$	7 **SR**-$\strictif''$
9	$\mathbf{v}(\Box P, \mathbf{w}_i) = \mathbf{T} \rightarrow \mathbf{v}(Q \strictif P), \mathbf{w}_i = \mathbf{T}$	2-8 $\rightarrow$ I
10	$\mathbf{Rww}_i \rightarrow (\mathbf{v}(\Box P, \mathbf{w}_i) = \mathbf{T} \rightarrow \mathbf{v}(Q \strictif P), \mathbf{w}_i = \mathbf{T})$	1-9 $\rightarrow$ I
11	$\mathbf{v}(\Box P \strictif (Q \strictif P), \mathbf{w}) = \mathbf{T}$	10 **SR**-$\strictif''$

3 Conclusion

This completes the exposition of basic semantical and derivational rules for our study of modal sentential logic. We shall now look at the system *K* which results from the use of the basic rules examined here. Later, we will turn to a number of other systems that result from strengthening the rules in various ways.