Module 3
Basic Syntax and Semantics for Modal Sentential Logic

G. J. Mattey

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Contents

In this module, we will begin our examination of *Modal Sentential Logic*, or *MSL*. Modal Sentential Logic is an *extension* of non-modal Sentential Logic. All the sentences of *SL* are sentences of *MSL*, and some sentences of *MSL* are not sentences of *SL*. The first section of the module will describe the syntax of *MSL*. The set of expressions of *SL* will be expanded by the addition of modal operators, and the set of sentences composed of those expressions will be expanded by the addition of sentences containing modal operators. The brief second section shows how each of the modal operators can be defined in terms of the others. The third and final section sets out the basic elements of semantics of *MSL*. In later modules, the basic semantics will be refined in order to produce various semantical systems of modal logic.

1 Syntax of Modal Sentential Logic

Historically, a number of different modal sentential languages have been constructed by taking different modal operators as primitive.[1] Most versions of modal logic syntax add to the expressions of *SL* two one-place operators: the box '$\Box$' and the diamond '$\Diamond$.'[2] We will treat these operators as primitive and add to the set of primitive operators the two-place "strict implication" operator, '$\prec$.' A sentence whose main logical operator is a modal operator will be called a *modal sentence*.[3]

1.1 Expressions of *MSL*

We extend the expressions of *SL* to obtain the vocabulary of *MSL*.

- An infinitely large set of *sentence letters* $A, B, C, \ldots, Z, A_1, B_1, \ldots, Z_1, A_2, B_2, \ldots$.
- A sentential constant '$\bot$.'
- Two punctuation marks '(' and ').'
- A set of five *truth-functional operators*, '$\sim$,' '$\wedge$,' '$\vee$,' '$\supset$,' and '$\equiv$.'
- A set of three *modal operators*, '$\Box$,' '$\Diamond$,' '$\prec$.'

Every expression of *SL* is an expression of *MSL*, but not every expression of *MSL* is an expression of *SL*.

1.2 Rules of Formation for *MSL*

Corresponding to the three new modal operators are three new rules of formation, rules 8 through 10.

1. All sentence letters are sentences of *MSL*.
2. '$\bot$' is a sentence of *MSL*.
3. If α is a sentence of *MSL*, then $\sim\alpha$ is a sentence of *MSL*.
4. If α and β are sentences of *MSL*, then $(\alpha \wedge \beta)$ is a sentence of *MSL*.

[1] An operator is *primitive* when it is specified in the original list of expressions of the vocabulary to which it belongs. An operator is *defined* when it does not appear in the original list, but is instead specified through the use of a definition. Examples will be given in the next section, beginning on page 3.

[2] Hughes and Cresswell in *A New Introduction to Modal Logic* use '*L*' and '*M*,' respectively.

[3] The main logical operator of a sentence is the operator featured in the last rule of formation that generates the sentence.

5. If α and β are sentences of *MSL*, then $(\alpha \vee \beta)$ is a sentence of *MSL*.
6. If α and β are sentences of *MSL*, then $(\alpha \supset \beta)$ is a sentence of *MSL*.
7. If α and β are sentences of *MSL*, then $(\alpha \equiv \beta)$ is a sentence of *MSL*.
8. If α is a sentence of *MSL*, then $\Box\alpha$ is a sentence of MSL.
9. If α is a sentence of *MSL*, then $\Diamond\alpha$ is a sentence of *MSL*.
10. If α and β are sentences of *MSL*, then $(\alpha \dashv\!\!\exists\ \beta)$ is a sentence of *MSL*.
11. Nothing else is a sentence of *MSL*.

We will stipulate that as an abbreviation, outermost parentheses may be omitted from any sentence after the application of the formation rules has been completed.

Since the formation rules for *MSL* include all those for *SL*, every sentence of *SL* is a sentence of *MSL*. But the *MSL* rules of formation allow the generation of sentences of *MSL* that are not sentences of *SL*. Thus, *MSL* is an extension of *SL*.

2 Informal Interpretations of *MSL* Operators

In the period before contemporary formal semantics was developed, modal operators were taken to represent modal terms in natural languages such as English.[4] This can be called their "intended interpretation." In themselves, the modal operators and sentences formed from them have no meaning, but since the development of possible-worlds semantics, they have been given precise meanings by formal semantical rules. Before introducing these rules, we will discuss the intended interpretations of the modal operators.

2.1 '□' as a Necessity Operator

In the most common informal interpretation, the box '□' is supposed to represent necessary truth.[5] For example, we might wish to represent the necessary truth of the truth-functionally valid *SL* sentence '$\sim(A \wedge \sim A)$' by affixing a box to it:

$\Box\sim(A \wedge \sim A)$.

2.2 '◊' as a Possibility Operator

The diamond generally is taken to represent possibility. A simple example is a case where it is affixed to the left of a sentence letter such as '*A*.' In the semantical system *SI*, all sentence letters are true on at least one interpretation, so we may wish to say that '*A*' is at least possibly true, which can be expressed as follows:

$\Diamond A$.

[4]Or they were interpreted using matrices, as described in Module 1.

[5]Analogous modal operators obeying exactly the same semantical rules have been used to represent knowledge, belief, obligation, future times, and other notions. These informal interpretations will be discussed extensively in Module 6 and following modules.

On a row of a truth-table on which 'A' is false, its negation, '$\sim A$,' is true. So we might say that the negation is possibly true, in which case 'A' is possibly false. We can state that 'A' is possibly true and possibly false in this way:

$\Diamond A \wedge \Diamond\sim A$.

In such a case, the sentence is interpreted in English as representing a state of affairs that is *contingent*.

2.3 '$\prec$' as a Strict Implication Operator

The two-place fish-hook operator is intended to represent "strict implication." This is a relation between two sentences that holds when it is impossible for the first to be true and the second false. When the truth-values are assigned as in the semantics for *SL*, this amounts to the same thing as saying that the first semantically entails the second. In *SL*, the sentence '$A \wedge B$' semantically entails 'A.' So we may wish to write:

$(A \wedge B) \prec A$.

In the next section, these informal notions will be given a rigorous treatment. But it can be seen already that the modal operators can be used to express in *MSL* some of the most important semantical properties and relations of *SL* sentences.[6] This was Lewis's original goal in developing modal logic.[7]

3 Inter-Definition of Modal Operators

The modal language *MSL* presented here contains a modest number of modal operators in its vocabulary. Each of the operators in the vocabulary of *MSL* is treated as *primitive* or undefined. The vocabulary could just as well have been stated using just one primitive modal operator. The other operators could then be defined in terms of that one primitive operator. Despite the fact that '$\Box$,' '$\Diamond$,' and '$\prec$' are primitive in *MSL*, we will take a look at how they could be defined in terms of one another. Then we will examine some other modal operators that can be defined.

3.1 Reduction to One Primitive Modal Operator

The definitions are motivated by the intended interpretations of the operators. Thus if we think of the '$\Box$' as expressing necessity, the '$\Diamond$' as expressing possibility, and the '$\sim$' as expressing negation, we might say that '$\Box A$' means that A is necessary. Given that what is necessary is not possibly not the case, it seems that '$\Box A$' should be taken as expressing the same thing as '$\sim\Diamond\sim A$.' It turns out that the two sentences are equivalent given the semantics to be developed later.

On the other hand, with the same intended interpretations, we might wish to say that '$\Diamond A$' expresses that same thing as '$\sim\Box\sim A$.' What is possible is not necessarily not the case. Again, these two sentences will proved to be equivalent in the semantical system we will develop.

The symbol for strict implication is intended to indicate the impossibility that the antecedent is true and the consequent false. Thus, $A \prec B$ could be defined as '$\sim\Diamond(A \wedge \sim B)$' or as '$\Box(A \supset B)$.' All three of these will prove to be semantically equivalent.

More generally, with '$\Diamond$' as primitive:

$\Box\alpha =_{\text{df}} \sim\Diamond\sim\alpha$.

[6] As will be seen, they can also express semantical properties and relations of *MSL* sentences.

[7] For example, "The strict implication, $p \prec q$, means 'It is impossible that p be true and q false . . . .' " *A Survey of Symbolic Logic*, pp. 332-333.

$\alpha \prec \beta =_{df} \sim\Diamond(\alpha \wedge \sim\beta)$.

With '$\Box$' as primitive:

$\Diamond\alpha =_{df} \sim\Box\sim\alpha$.

$\alpha \prec \beta =_{df} \Box(\alpha \supset \beta)$.

We could also take the fish-hook as a primitive operator, defining the box and the diamond in terms of it.

Exercise. Define the two primitive one-place operators in terms of the fish-hook [challenging].

3.2 Other Modal Operators

A number of other modal operators may be defined in *MSL* or taken as primitive.

3.2.1 Intensional Disjunction

In his earliest papers on modal logic, written in 1912, Lewis worked with an operator he called "intensional" or "dilemmatic" disjunction.[8] An intensional disjunction $\alpha \veebar \beta$ is true just in case the falsehood of one of the disjuncts logically (or "strictly") implies the truth of the other.[9] Another way Lewis put it is that it is impossible for both disjuncts to be false.[10] Lewis's stock example of an intensional disjunction was "Either Matilda does not love me or I am beloved." We may symbolize this sentence in non-modal Predicate Logic using a one-place predicate '*Lx*,' for 'x loves me' and the letter '*m*' for 'Matilda.' Then we have '$\sim Lm \vee (\exists x)Lx$.' The falsehood of '$\sim Lm$' implies the truth of '*Lm*,' which in turn implies the truth of '$(\exists x)Lx$.' The falsehood of '$(\exists x)Lx$' implies the truth of '$\sim Lm$.'

This account of intensional disjunction suggests the following definition in terms of the strict implication operator, which Lewis in fact adopted:

$\alpha \veebar \beta =_{df} \sim\alpha \prec \beta$.

We can also define $\alpha \veebar \beta$ in terms of the other two primitive operators of the language *MSL*:

$\alpha \veebar \beta =_{df} \sim\Diamond(\sim\alpha \wedge \sim\beta)$.

$\alpha \veebar \beta =_{df} \Box(\alpha \vee \beta)$.

The reader might notice that the first definition parallels precisely the definition in non-modal Sentential Logic of '$\vee$' in terms of '$\sim$' and '$\supset$.' The definition of '$\supset$' in terms of '$\sim$' and '$\vee$' is paralleled by the following definition:

$\alpha \prec \beta =_{df} \sim\alpha \veebar \beta$.

3.2.2 Impossibility

By 1914, Lewis employed a one-place primitive operator '$\sim$,' which was intended to express impossibility.[11] From this starting point, we can give the following definitions:

[8] "Implication and the Algebra of Logic" and "A New Algebra of Implications."

[9] The symbol '$\veebar$' is peculiar to this text. There is no standard symbol for intensional disjunction.

[10] "The Calculus of Strict Implication," 1914.

[11] "The Matrix Algebra for Implications," as well as in *A Survey of Symbolic Logic*, 1918. Note that the symbol in the present type-face is slightly thicker than the symbol for negation.

$\Diamond\alpha =_{\text{df}} \sim\sim\alpha.$

$\Box\alpha =_{\text{df}} \sim\sim\alpha.$

$\alpha \strictif \beta =_{\text{df}} \sim(\alpha \wedge \sim\beta).$

$\alpha \veebar \beta =_{\text{df}} \sim(\sim\alpha \wedge \sim\beta).$

3.2.3 Consistency

At the same time Lewis introduced the impossibility operator, he introduced a defined operator for consistency: '$\circ$.' Two sentences α and β are said to be consistent just in case it is possible that they both be true. Consistency can be defined in terms of other operators as follows:

$\alpha \circ \beta =_{\text{df}} \Diamond(\alpha \wedge \beta).$

$\alpha \circ \beta =_{\text{df}} \sim\Box\sim(\alpha \wedge \beta).$

$\alpha \circ \beta =_{\text{df}} \sim(\alpha \strictif \sim\beta).$

$\alpha \circ \beta =_{\text{df}} \sim(\alpha \veebar \beta).$

$\alpha \circ \beta =_{\text{df}} \sim\sim(\alpha \wedge \beta).$

3.2.4 Strict Equivalence

The final modal operator introduced by Lewis was strict equivalence, symbolized here by '$\strictiff$.'[12] We shall here only give the definition of strict equivalence in terms of the three primitive modal operators of this text:

$\alpha \strictiff \beta =_{\text{df}} \sim\Diamond(\alpha \wedge \sim\beta) \wedge \sim\Diamond(\beta \wedge \sim\alpha).$

$\alpha \strictiff \beta =_{\text{df}} \Box(\alpha \supset \beta) \wedge \Box(\beta \supset \alpha).$

$\alpha \strictiff \beta =_{\text{df}} (\alpha \strictif \beta) \wedge (\beta \strictif \alpha).$

Strict equivalence is the only one of the operators, primitive or defined, that cannot be made primitive without some loss of expressive power. This is analogous to a result for *SL*, according to which the material biconditional '$\equiv$' (even with '$\sim$') is not sufficient to define the other non-modal operators.[13]

Exercise. Given the informal interpretations of the operators in question, show how to define the fish-hook in terms of the circle.

4 Basic Semantics for *MSL*

The semantics for Modal Sentential Logic is a generalization of the semantics for non-modal Sentential Logic. The semantical rules of *SI* assign a truth-value to a non-atomic sentence based on the truth-value(s) of its component(s). From the truth-value assignment (TVA) made to the sentence letters by the interpretation, the truth-value of the sentence results immediately from the application of the semantical rules. Thus if an interpretation assigns to the sentence-letter 'A' the value **T**, then the value of '$\sim A$' is **F**, etc. In the last

[12]Lewis's symbol was '='. He used that symbol for both strict equivalence and definitional identity, which we would now say is mixing the object-language with the meta-language.

[13]See Geoffrey Hunter, *Metalogic*, 1996, p. 69.

module, it was proved that the operators and their semantical rules of *SI* are truth-functional. We will now say that in such a case, the operators and the semantical rule governing them are *directly* truth-functional.

For modal sentences (sentences whose main operator is modal), the situation is different: the standard semantical rules for modal operators are *not* directly truth-functional. We cannot obtain the truth-value of '$\Box A$' on an interpretation simply from a single value for 'A' resulting from the TVA made by the interpretation. The semantical rules for sentences whose main operator is a one-place modal operator are not directly truth-functional in the way the one-place negation operator is. Nonetheless, the determination of the value of the modal sentence '$\Box A$' will turn out to be a function of (possibly) more than one truth-value assigned by the given interpretation to 'A,' and so semantics for modal sentential logic may be said to be *indirectly* truth-functional in a way to be described fully below.

4.1 Truth-Value Assignments

We may depict in the standard truth-table format a partial TVA for an *SL* sentence. So, for example, if an interpretation I assigns **T** to 'A' and **F** to 'B,' we have a table that looks like this.

A	B
T	**F**

And by **SR**-$\wedge$ we get:

A	B	$A \wedge B$
T	**F**	**F**

where the value of the conjunction is **F** because the assignment to one of its conjuncts is **F**.

Note that this determination is entirely self-contained. No reference is made to any other TVA but the one which assigns **T** to 'A' and **F** to 'B.' To make another assignment would be to give a different interpretation, since there is no distinction in *SI* between an interpretation and the truth-value assignments it makes. The truth and falsehood of a sentence is relative to a single given truth-value assignment, and hence to a single given interpretation.

A different interpretation with a different TVA gives a different result for the sentence '$A \wedge B$.'

A	B	$A \wedge B$
T	**T**	**T**

Once again, no reference is made to any other truth-value assignment. It is as if each TVA represents a "world" of its own. In terms of the formal semantics, we might say that there are (at least) two interpretations, I′ and I″, such that $v_{I'}(A)$=**T** and $v_{I'}(B)$=**F**, while $v_{I''}(A)$=**T** and $v_{I''}(B)$=**T**. Then $v_{I'}(A \wedge B)$=**F** and $v_{I''}(A \wedge B)$=**T**.

Although the determination of the truth-values of sentences of *SL* depends entirely on an interpretation and its TVA, other semantic properties of the sentence can be determined only by looking at more than one interpretation of it. Whether a sentence is truth-functionally valid, for example, requires that we look at the values for the sentence under different TVAs. Thus the sentence '$A \wedge B$' is truth-functionally indeterminate, as we have already shown by producing a TVA under which it is true and a different one in which it is false.[14]

[14]A sentence is *truth-functionally indeterminate* just in case there is an interpretation that assigns it the value **T** and an interpretation that assigns it the value **F**. All sentence letters are truth-functionally indeterminate.

Now let us consider the disjunction '$A \vee {\sim}A$,' which is truth-functionally valid. We can say this because there are exactly two possible partial TVAs making an assignment to 'A,' producing the following two tables.[15]

A	$A \vee {\sim}A$
T	**T**

A	$A \vee {\sim}A$
F	**T**

Granting that the assignments made to any other sentence letters are irrelevant to the truth or falsehood of '$A \vee {\sim}A$,' we can say that it is true on all interpretations (which just are truth-value assignments), and so it is truth-functionally valid.

Truth-functional validity and the related notions are concepts which are not expressible in the syntax of *SL* itself, but only in the meta-language we use to talk about *SL*. The formal meta-linguistic notation for truth-functional truth,

$\vDash_{SI} A \vee {\sim}A$,

uses a symbol that is not included in the syntax of *SL*. But we can express something analogous in *MSL*:

$\Box(A \vee {\sim}A)$.

One way to understand the modal operators is as simulating, in the modal object-language *MSL*, meta-logical properties and relations. This is done by extending the object-language *SL* to include modal sentences *whose truth-values on an interpretation can depend on multiple truth-value assignments*. Such an extension allows the representation in the object language *MSL* of semantical properties and relations that are more general than simple truth and falsehood.

Though there are three modal operators defined in the syntax of *MSL*, we shall limit our initial discussion to the one-place modal operators '$\Box$' and '$\Diamond$.' From a given sentence α we can form the necessity-sentence $\Box\alpha$ in *MSL*. We will express in the formal semantics the informal meaning of the necessity sentence according to which $\Box\alpha$ is true just in case α is necessarily true. A possibility-sentence $\Diamond\alpha$ will be true in the formal semantics just in case it is possible that α is true. (Again, it must be stressed that these modal operators can be given other readings.)

From a formal standpoint, to interpret necessity- and possibility-sentences we need a way of making reference to multiple truth-value assignments *within a single interpretation*. The semantics for Modal Sentential Logic does not identify interpretations with TVAs in the manner of the semantics for non-modal Sentential Logic. In the semantics for *MSL*, a single interpretation must be able to allow more than one distinct truth-value assignment to the sentence letters. This means that the application of an *SL* valuation-function to a sentence α, $\mathbf{v_I}(\alpha)$, is not adequate for the semantics for *MSL*.

To accommodate multiple truth-value assignments, we require what are most commonly known as "*possible worlds*."[16] We may think of possible worlds as "locations" with respect to which truth-value assignments are made. Arbitrary worlds will be indicated by the meta-variable '**w**' with or without primes or a lower-case italic alphabetic subscript. Specific worlds will be indicated by 'w' with or without primes or

[15] Of course, this result is usually depicted in a single table, but two are used here to indicate the fact that it is not a row on a truth-table by itself, but the matching of a truth-value to a sentence, which gives the desired result.

[16] They have also been called assignments, cases, situations, states of affairs, indices, points, etc. Whatever their name, they are nothing more than reference points for assignments of truth-values to sentences.

positive integer subscripts. We shall have recourse to alphabetic subscripts, such as with '$\mathbf{w}_i$,' when we wish to talk about arbitrary worlds, as will be done below.

Thus, to build on our previous example, we could say that a specific interpretation I of '*A*' contains two partial truth-value assignments. We can call the first partial TVA, which assigned '*A*' the value **T**, an *assignment at* w_1, and the second partial TVA, which assigned '*A*' the value **F**, an *assignment at* w_2.

w_1

A	$A \vee {\sim}A$
T	**T**

w_2

A	$A \vee {\sim}A$
F	**T**

4.2 Formal Semantics for *MSL*

As with Sentential Logic, we will give a formal specification of the basic semantics for modal logic. In later modules, we shall make good use of the formal semantics given here.

4.2.1 Generalized Valuation-Functions

Now we can extend the notion of a valuation-function to make it adequate for the formal semantics of *MSL*. What is required is a two-place function $\mathbf{v_I}$, which maps a pair consisting of a sentence *and a world* onto truth-values, i.e., the set {**T**,**F**}. In the example just given, we can say with respect to our interpretation I and its valuation-function v:

$v_I(A, w_1)=\mathbf{T}$,

$v_I(A, w_2)=\mathbf{F}$.

It is easy to see by **SR**-∼ and **SR**-∨, that for any world **w**,

$v_I(A \vee {\sim}A, \mathbf{w})=\mathbf{T}$.

(It might be helpful to recognize that we could have used two-place valuation functions in the semantics for sentential logic. That is, instead of subscripting the '**v**' with a reference to the interpretation under which it makes its assignments, we could have made that reference the second argument of the function:

$\mathbf{v_I}(\alpha) = \mathbf{v}(\alpha, \mathbf{I})$.

But this notation would not have allowed the generalization to the semantics for modal logic, since it presupposes a one-to-one correspondence between interpretations and truth-value assignments.)

4.2.2 The Accessibility Relation

At this point, we are close to being able to state the specific semantical rules for the determination of the truth-values of necessity- and possibility-sentences at a world. But we require one more piece of machinery. We must think of the possible worlds in an interpretation as ordered under a relation of *accessibility*. The inclusion of this relation in the semantics for modal logic is the hallmark of modern "possible worlds" semantics.

We will say, for example, that on an interpretation I, a specific world, w_2, is accessible to a specific world, w_1. This can be depicted graphically with an arrow from w_1 to w_2.[17]

$$w_1 \longrightarrow w_2$$

The meta-variable **R** will indicate an arbitrary two-place relation of accessibility. In keeping with our general practice, we shall use 'R' for a specific accessibility relation. Thus we can state what was just depicted graphically in symbolic terms as: Rw_1w_2. Every modal logic interpretation requires both a non-empty set of worlds and a relation of accessibility among the members of the set of worlds. (Unless restrictions are placed on the accessibility relation, any world in a given interpretation may or may not be accessible to any other.)

Together, the set of worlds and the accessibility relation constitute a *frame*. If we use '**W**' to indicate an arbitrary set of worlds, a frame **Fr** can be represented as an ordered pair, so that **Fr**=⟨**W**,**R**⟩.[18] We add a valuation-function '**v**' to a frame to get an interpretation: **I**=⟨**W**,**R**,**v**⟩. The interpretation formed in this way is said to be *based on* the frame to which the valuation function is added. We will say that a world is *in* an interpretation **I** when it is a member of the set **W**, which itself is a member of **I**.

We can express a two-place relation as a set of ordered pairs, the first of the two bearing the relation to the second. In the case of accessibility, the first member of a pair will be a world and the second member a world accessible to the first world. An example of a frame is:

$\langle\{w_1, w_2, w_3\}, \{\langle w_1 w_2\rangle, \langle w_2 w_3\rangle, \langle w_3 w_1\rangle\}\rangle$.

This may be represented graphically as follows:

$$\begin{array}{c}\longleftarrow\\ w_1 \longrightarrow w_2 \longrightarrow w_3\end{array}$$

Insofar as we think of a possible world as representing a row of a truth-table, we can think of each world in a frame as a row of a truth-table which does not have its values filled in. Suppose we have a frame with two worlds, w_1 and w_2, such that Rw_1w_2. A partial representation of the frame, covering only the sentence letters '*A*' and '*B*,' might look like the following.

w_1		$\longrightarrow$	w_2	
A	*B*		*A*	*B*
?	?		?	?

An interpretation would give truth-values to *A* and *B*. One such interpretation, which assigns **T** to *A* and **F** to *B* at w_1 and **T** to both *A* and *B* at w_2, can be represented as follows.

w_1		$\longrightarrow$	w_2	
A	*B*		*A*	*B*
T	**F**		**T**	**T**

[17] Hughes and Cresswell use the metaphor of being able to "see" w_2 from w_1.

[18] Ordered *n*-tuples are expressed with angled brackets and *n* elements, separated by commas.

The accessibility relation is the basis for determining the truth-value of modal sentences. Specifically, a sentence of the form $\Diamond\alpha$ is true at a world if and only if it is true at some accessible world, and a sentence of the form $\Box\alpha$ is true at a world if and only if it is true at all accessible worlds. In the example just given, the following values would be generated.

$$w_1 \longrightarrow w_2$$

A	B		A	B
T	**F**		**T**	**T**

$\Box A$	$\Diamond B$
T	**F**

In the case of $\Box\alpha$, since there is only one world accessible to w_1, and α is true there, $\Box\alpha$ is true at w_1. There is no world accessible to w_2 at which B is true, and so $\Diamond B$ is false at w_2.

In some cases, accessibility plays no role in the evaluation of modal sentences. If, as with $\alpha \vee {\sim}\alpha$, there is no way to give the sentence the value **F** at any world, it will have the value **T** at all worlds that could be accessible to a given world. So $\Box(\alpha \vee {\sim}\alpha)$ will have the value **T** at all worlds.

The examples given here have been simple illustrations of the basic semantics. We will now turn to a formal specification of the semantics.

4.2.3 Semantical Rules

Because valuation-functions have been generalized to include a second argument referring to possible worlds, we will have to re-state the semantical rules for the truth-functional operators. All the rules for those operators operate locally, so to speak, in that they apply only at a specific possible world. To the generalization of the truth-functional rules we add the semantical rules (or truth-definitions) for sentences governed by the modal operators.

SR-TVA If α is a sentence-letter, then either $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{T}$ or $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{F}$, but not both.

SR-⊥ For all **I** and all **w** in **I**, $\mathbf{v_I}(\bot, \mathbf{w})=\mathbf{F}$, and $\mathbf{v_I}(\bot, \mathbf{w})\neq\mathbf{T}$.

SR-∼ $\mathbf{v_I}({\sim}\alpha, \mathbf{w})=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{F}$; $\mathbf{v_I}({\sim}\alpha, \mathbf{w})=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{T}$.

SR-∧ $\mathbf{v_I}(\alpha \wedge \beta, \mathbf{w})=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{T}$ and $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{T}$; $\mathbf{v_I}(\alpha \wedge \beta, \mathbf{w})=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{F}$ or $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{F}$.

SR-∨ $\mathbf{v_I}(\alpha \vee \beta, \mathbf{w})=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{T}$ or $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{T}$; $\mathbf{v_I}(\alpha \vee \beta, \mathbf{w})=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{F}$ and $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{F}$.

SR-⊃ $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w})=\mathbf{T}$ if and only if either $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{F}$ or $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{T}$; $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w})=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{T}$ and $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{F}$.

SR-≡ $\mathbf{v_I}(\alpha \equiv \beta, \mathbf{w})=\mathbf{T}$ if and only if either $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{T}$ and $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{T}$, or $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{F}$ and $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{F}$; $\mathbf{v_I}(\alpha \equiv \beta, \mathbf{w})=\mathbf{F}$ if and only if either $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{T}$ and $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{F}$, or $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{F}$ and $\mathbf{v_I}(\beta, \mathbf{w})=\mathbf{T}$.

SR-◊ $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)=\mathbf{T}$ at some world $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$; $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)=\mathbf{F}$ at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$.

SR-□ $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**T** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$; $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**F** if and only if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**F** at some world $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$.

SR-⥽ $\mathbf{v_I}(\alpha \mathbin{⥽} \beta, \mathbf{w})$=**T** if and only if either $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**F** or $\mathbf{v_I}(\beta, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$; $\mathbf{v_I}(\alpha \mathbin{⥽} \beta, \mathbf{w})$=**F** if and only if both $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** and $\mathbf{v_I}(\beta, \mathbf{w}_i)$=**F** at some world $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$.

These semantical rules and definitions of ‘frame’ and ‘interpretation,’ when added to the semantical rules for Sentential Logic, yield a semantical system for modal logic that we will for the time being call the “basic” semantical system.

We will illustrate the use of the semantical system *KI* with an example. Consider an interpretation I where W = {w_1, w_2}, Rw_1w_1, Rw_1w_2, and $v_I(A, w_1)$ = **T**, $v_I(A, w_2)$ = **T**.

↶		
w_1	⟶	w_2
A		A
T		**T**

On such an interpretation, it follows that $v_I(\Box A, w_1)$ = **T**. The reason is that worlds w_1 and w_2 are all the worlds accessible to w_1, and ‘A’ is true at both of them. The value of ‘$\Box A$’ at w_2 is not as straightforwardly determined, because there are no worlds accessible to w_2. It is standard practice to say that in this case, it is *vacuously* the case that ‘$\Box A$’ is true at w_2. (That is, **SR**-□ is understood as saying that if there are any accessible worlds, then α is true at such worlds. Since the antecedent of the conditional is false in the present instance, the conditional itself is taken as true. The meta-logical conditional implicit in **SR**-□ is treated as a material conditional!) Therefore, $v_I(\Box A, w_2)$ = **T**.

↶		
w_1	⟶	w_2
A		A
T		**T**
$\Box A$		$\Box A$
T		**T**

Further, $v_I(\Diamond A, w_1)$ = **T**, since there is an accessible world (w_1 as well as w_2) at which ‘A’ is true. However, $v_I(\Diamond A, w_2)$ = **F**. There is no world accessible to w_2 at which ‘A’ is true.

↶		
w_1	⟶	w_2
A		A
T		**T**
$\Diamond A$		$\Diamond A$
T		**F**

This shows the weakness of the basic semantics we have developed. There are interpretations on which a sentence is “necessarily true” but not “possibly true.” This is because the basic semantics allows what Hughes and Cresswell call “dead-end” worlds, worlds to which no world is accessible.

4.3 Meta-Logical Properties and Relations in *MSL*

Given the truth-definitions for sentences of *MSL*, we can define semantical properties and relations of *MSL* sentences analogous to those of *SL* sentences. Because different restrictions on the accessibility relation in frames will generate different systems of modal logic, we shall here give formal definitions which are fully general and can apply to any semantical system which will be considered in what follows. For this reason, we will define the notions of semantical entailment and validity relative to a frame. We can then later define them for more specific modal systems. We shall postpone the proof of modal forms Bivalence and Truth-Functionality, as well as discussion of a modal form of Semantical Consistency, until Module 5, when we consider the semantical system *KI*.

4.3.1 Semantical Entailment in a Frame

Modal semantics requires that we expand the definition of semantical entailment to accomodate the inclusion in the semantics of possible worlds, the accessibility relation, and the two-place valuation-function. In the modal semantics, a sentence only has a truth-value at a world. Thus we will say that relative to a given frame, the relation of semantic entailment holds between a set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ and a sentence α when all the interpretations based on the frame which make all the sentences γ_i of the set true at a world also make the sentence α true at that world. The strict definition of semantical entailment in a frame **Fr** is as follows:[19]

Semantical Entailment in a Frame Fr

$\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \vDash_{\mathbf{Fr}} \alpha$ just in case for any **I** based on **Fr** and any **w** in **W** in **Fr**, if $\mathbf{v_I}(\gamma_1, \mathbf{w})$=**T**, $\mathbf{v_I}(\gamma_2, \mathbf{w})$=**T**, and $\cdots$, and $\mathbf{v_I}(\gamma_n, \mathbf{w})$=**T**, then $\mathbf{v_I}(\alpha, \mathbf{w})$=**T**.

For example, consider the frame used earlier. Fr = $\{\langle\{w_1, w_2\}, \{\langle w_1, w_1\rangle, \langle w_1, w_2\rangle\}$. Suppose we want to determine whether $\{\Box A\} \vDash_{\mathrm{Fr}} \Diamond A$. That is, we want to determine whether there are any interpretations based on Fr such that at some world in W, '$\Box A$' is true while '$\Diamond A$' is false. In fact, there is such an interpretation, the one we considered above. It contains a world, w_2, such that v($\Box A$, w_2)=**T** and v($\Diamond A$, w_2)=**F**.

w_1		$\longrightarrow$	w_2	
A	B		A	B
T	**F**		**T**	**T**
$\Box A$			$\Diamond A$	
T			**F**	

So the semantic entailment fails.

Now we will consider a semantical entailment which holds for all frames: $\{\Box(A \wedge B)\}$ entails $\Box A$. Suppose for an arbitrary frame **Fr**, an arbitrary world **w** in **Fr**, and an arbitrary interpretation *I* based on **Fr**, that $\mathbf{v_I}(\Box(A \wedge B), \mathbf{w} = \mathbf{T}$. Then by **SR-□**, at all accessible worlds $\mathbf{w}_i$, $\mathbf{v_I}(A \wedge B), \mathbf{w}_i = \mathbf{T}$. By **SR-∧**, $\mathbf{v_I}(A, \mathbf{w}_i) = \mathbf{T}$. Then again by **SR-□**, $\mathbf{v_I}(\Box A, \mathbf{w}) = \mathbf{T}$.

We may illustrate this reasoning by way of truth-tables. To do so, we need a further notational device. When we place an asterisk over an accessibility arrow, it indicates that we are reasoning about all accessible worlds. (When an asterisk is placed below the arrow, it indicates that we are reasoning about at least one accessible world.) The first step is to assume that $\Box(A \wedge B)$ is true at an arbitrary world **w** and infer that $A \wedge B$ is true at all accessible worlds $\mathbf{w}_i$.

[19] In keeping with our standard practice, a specific frame will be indicated by the non-bold 'Fr.'

$\mathbf{w}$	$\xrightarrow{*}$	$\mathbf{w}_i$
$\Box(A \wedge B)$		$A \wedge B$
T		**T**

Next we infer that the truth-value of $A \wedge B$ at $\mathbf{w}_i$ is **T**.

$\mathbf{w}$	$\xrightarrow{*}$	$\mathbf{w}_i$
$\Box(A \wedge B)$		$A \wedge B$
T		**T**
		A
		T

Finally, we infer that $\Box A$ is true at $\mathbf{w}$.

$\mathbf{w}$	$\xrightarrow{*}$	$\mathbf{w}_i$
$\Box(A \wedge B)$		$A \wedge B$
T		**T**
		A
		T
$\Box A$		
T		

4.3.2 Semantical Equivalence in a Frame

We can use the definition of entailment in a frame to form a definition of *equivalence in a frame*. Two sentences of *MSL* α and β are equivalent in a frame **Fr** just in case the set consisting of the one semantically entails in the frame the other, and vice-versa.

Semantical Equivalence in a Frame Fr

α is semantically equivalent to β in a frame **Fr** if and only if for all interpretations **I** based on **Fr** and all worlds **w** in **FR**, $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{v_I}(\beta, \mathbf{w})$.

The following meta-theorem can then be proved.

α is semantically equivalent to β in frame **Fr** just in case $\{\alpha\} \vDash_{\mathbf{Fr}} \beta$ and $\{\beta\} \vDash_{\mathbf{Fr}} \alpha$.

Exercise. Prove the meta-theorem just stated.

We will here illustrate with truth-tables the semantical equivalence of '$\sim\Box A$' and '$\Diamond\sim A$.'

$\mathbf{w}$	$\xrightarrow{*}$	$\mathbf{w}_i$
$\Box\sim A$		
T		
		$\sim A$
		T
		A
		F
$\Diamond A$		
F		
$\sim\Diamond A$		
T		

The following table reflects the fact that if a possibility-sentence $\Diamond\alpha$ is false at a world, then α is false at all accessible worlds, by **SR-**$\Diamond$.

$\mathbf{w}$	$\xrightarrow{*}$	$\mathbf{w}_i$
$\sim\Diamond A$		
T		
$\Diamond A$		
F		
		A
		F
		$\sim A$
		T
$\Box\sim A$		
T		

4.3.3 Validity in a Frame

The limiting case of entailment in a frame is *validity in a frame*. We have seen in the case of Sentential Logic that validity can be treated as semantic entailment by the empty set of sentences, and here it is understood as relativized to a frame: $\emptyset \vDash_{\mathbf{Fr}} \alpha$. A sentence α is valid in a frame if and only if it is true at all worlds on all interpretations based on that frame.

Validity in Frame Fr

$\vDash_{\mathbf{Fr}} \alpha$ iff for all **w** in **W** in **Fr**, all **I** based on **Fr**, and all $\mathbf{v_I}$ in **I**, $\mathbf{v_I}(\alpha, \mathbf{w})$=**T**

To return to a previous example, '$\Box(A \lor \sim A)$' is valid in any frame whatsoever. No matter how 'A' is interpreted at any given world, as **T** or **F** at that world, the sentence '$A \lor \sim A$' is true at every world that could be in a frame. Therefore, '$A \lor \sim A$' is true at any world accessible to a given world **w** in any frame, in which case '$\Box(A \lor \sim A)$' is true at **w**. So on any interpretation and any world based on any frame, '$\Box(A \lor \sim A)$' is true, which was to be proved. We can generalize this reasoning to conclude that the necessitation of every *SI*-valid sentence of *SL* is valid in every frame in the basic semantical system.

$\mathbf{w}$	$\xrightarrow{*}$	$\mathbf{w}_i$
		$A \lor \sim A$
		T
$\Box(A \lor \sim A)$		
T		

When entailment, equivalence, or validity holds for all frames, we shall drop the reference to the frame. Later, we will develop a number of semantical systems by placing restrictions on the accessibility conditions. Then we will introduce the notion of a *class of frames* containing the restricted accessibility relation as a member. Accordingly, we will speak of entailment, equivalence, and validity in all frames of a certain class.

5 Conclusion

It may be useful to reflect on the nature of the accessibility relation. One way to think of it is as representing those situations that matter with respect to the modality in question. For example, we might think of something as necessary when it is inevitable in the sense that there is no alternative that matters to its being the case. The alternatives that matter might be thought of as being "accessible worlds." Truth of the necessity-sentence $\Box\alpha$ would require truth of the embedded sentence α in all those worlds that matter. A possibility-sentence would indicate that its embedded sentence is true in at least one of the worlds that matters. We shall in what follows have much more to say about how accessibility captures our intuitions about the various kinds of modalities.

We are now finally in a position to see why the semantics for modal sentential logic is indirectly truth-functional. Although the truth of a necessity sentence (or any other modal sentence) is not directly a function of the assignments made to its components, it is a function of the assignments made to its components at the accessible possible worlds. And those assignments are ultimately based on truth-value assignments to the sentence-letters from which the sentence is composed. In Module 5, we will give a definition of modal Truth-Functionality in connection with specific semantical system and prove inductively that the system conforms to that definition.