Module 5
K and Equivalent Systems

G. J. Mattey

July 8, 2010

Contents

The weakest of the axiomatic systems to be considered in this text is the system *K* (named after Kripke).[1] The "basic" semantical and derivational rules spelled out in the last two modules are equivalent to *K*, in that parallel semantical properties and relations hold for them, as will be noted at the appropriate time. We will treat the semantical and derivational systems before turning to *K* proper. In subsequent modules, we shall examine systems stronger than *K*, i.e., systems which incorporate all the elements of *K* and support consequence-relations not found in *K* and its equivalent counterparts.

1 The Semantical System *KI*

Recall that the basic semantics for Modal Sentential Logic distinguishes between a frame and an interpretation based on a frame. A frame **Fr** contains a set **W** of possible worlds and a relation **R** of accessibility defined on those worlds: **Fr** = ⟨**W**,**R**⟩. An interpretation **I** based on a frame adds to it a valuation function **v** from sentences of *MSL* and worlds to truth-values, so that **I**=⟨**W**,**R**,**v**⟩. (We indicate the containment of **v** in an interpretation **I** by the notation '$\mathbf{v_I}$.') The semantical rules for the basic semantics for modal sentential logic define the behavior of the **v** function. The only difference between interpretations based on a given frame is the assignments made by **v** to the sentence letters at worlds and the resulting truth-values for compound sentences at worlds.

1.1 Specification of *KI*

The only restriction on **W** in a given frame is that there be at least one world in it. There are no restrictions on the accessibility relation **R**. The presence of **R** in a frame does not guarantee that any world stands in that relation to any world. In some frames, **R** is an empty relation. An *unrestricted class* of frames is a set of frames with no contraints on the accessibility relation. The set of all frames is an unrestricted class of frames. A *KI-frame* is defined as any member of the set of all frames. (In other words, every frame is a *KI*-frame.) It follows that the accessibility relation is unrestricted relative to the set of all *KI*-frames. A *KI-interpretation* is an interpretation based on a *KI*-frame, in which **v** obeys the basic semantical rules for modal sentential logic. The semantical system *KI* consists in the specification of a *KI* interpretation.

1.2 Semantical Properties and Relations in *KI*

We begin our treatment of the semantical properties and relations that hold in the system *KI* by re-visiting the properties of Bivalence and Truth-Functionality which were demonstrated to hold in *SI*. Next, we define the notions of Semantical Entailment, Semantical Equivalence, Validity, and Semantical Consistency relative to *KI*. We will then introduce a new semantical property, Closure under the '□,' which is specific to *KI* and all the stronger semantical systems we will consider in later modules.

1.2.1 Modal Bivalence

In Module 2, it was proved by mathematical induction that the non-modal semantical system *SI* has the propery of Bivalence: on every interpretation **I**, every sentence α of *SL* has the value **T** or the value **F**. Due to the additional components of the semantical system *KI*, the definition of Bivalence must be modified if it is to be applied to *KI*. Specifically, a reference to truth-value at a world must be built into the definition. Thus, we will say that on every interpretation **I**, and at every world **w** in **I**, the truth-value of a given sentence α at **w** is either **T** or **F**. We will call the extended sense of Bivalence 'Modal Bivalence' or '**MBV**' for short.

[1]A system is said to be weaker than another when there are consequence relations that hold in the stronger system but not in the weaker system.

MBV. $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{F}$.

The proof parallels that for non-modal sentential logic. The cases for sentences with no operators or whose main operators are non-modal operators are trivially modified to make $\mathbf{v_I}$ a two-place function. Three new cases are added for modal sentences.

Proof of Modal Bivalence by mathematical induction on the number n of operators in α.

Basis Step. $n = 0$. **Case 1**. α is a sentence letter. By **SR-TVA**, every sentence letter has at least one of the values **T** and **F** at each world on all interpretations. **Case 2**. α is $\bot$. Since $\mathbf{v_I}(\bot, \mathbf{w}) = \mathbf{F}$ at all worlds **w** on all interpretations, $\bot$ has at least one of the values **T** and **F** at all worlds **w** on all interpretations.

Inductive Hypothesis. Suppose that for sentences α of *MSL* with fewer than n operators, $\mathbf{w}_i$ in **I**, $\mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{F}$ on all interpretations **I** and at all worlds.

Induction Step. Let α contain n operators. It is to be proved that $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{F}$.

Case 1. α is $\sim\beta$.

1	$\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{F}$	Inductive hypothesis
2	| $\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}$	Assumption
3	| $\mathbf{v_I}(\sim\beta, \mathbf{w}) = \mathbf{F}$	1 **SR-~**
4	| $\mathbf{v_I}(\sim\beta, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\sim\beta, \mathbf{w}) = \mathbf{F}$	3 $\bigvee$ I
5	| $\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{F}$	Assumption
6	| $\mathbf{v_I}(\sim\beta, \mathbf{w}) = \mathbf{T}$	5 **SR-~**
7	| $\mathbf{v_I}(\sim\beta, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\sim\beta, \mathbf{w}) = \mathbf{F}$	6 $\bigvee$ I
8	$\mathbf{v_I}(\sim\beta, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\sim\beta, \mathbf{w}) = \mathbf{F}$	1 2-4 5-7 $\bigvee$ E

Case 2. α is $\beta \wedge \gamma$.

Case 3. α is $\beta \vee \gamma$.

Case 4. α is $\beta \supset \gamma$.

Case 5. α is $\beta \equiv \gamma$.

Case 6. α is $\Diamond\beta$. Given that Excluded Middle holds at the meta-logical level, there are exactly two kinds of interpretations **I**: At some accessible world $\mathbf{w}_i$, $\mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T}$; it is not the case that at some accessible world $\mathbf{w}_i$, $\mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T}$. It will be shown that on either kind of interpretation, a sentence of the form $\Diamond\beta$ has the value **T** or the value **F** at an arbitrary world, in which case it has the value **T** or **F** at all worlds on all interpretations.

1	$(\Pi\mathbf{w}_i)(\mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{T} \bigvee \mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{F})$	Inductive Hypothesis
2	$(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{T}) \bigvee \neg(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{T})$	Logic
3	$\quad(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{T})$	Assumption
4	$\quad\mathbf{v_I}(\Diamond\beta,\mathbf{w}) = \mathbf{T}$	3 **SR**-$\Diamond$
5	$\quad\mathbf{v_I}(\Diamond\beta,\mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\Diamond\beta,\mathbf{w}) = \mathbf{F}$	4 $\bigvee$ I
6	$\quad\neg(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{T})$	Assumption
7	$\quad\quad\mathbf{Rww}_1$	Assumption
8	$\quad\quad\quad\neg\mathbf{v_I}(\beta,\mathbf{w}_1) = \mathbf{F}$	Assumption
9	$\quad\quad\quad\mathbf{v_I}(\beta,\mathbf{w}_1) = \mathbf{T} \bigvee \mathbf{v_I}(\beta,\mathbf{w}_1) = \mathbf{F}$	1 Π E
10	$\quad\quad\quad\mathbf{v_I}(\beta,\mathbf{w}_1) = \mathbf{T}$	8 9 Disjunctive Syllogism
11	$\quad\quad\quad\mathbf{Rww}_1 \bigwedge \mathbf{v_I}(\beta,\mathbf{w}_1) = \mathbf{T}$	7 10 $\bigwedge$ I
12	$\quad\quad\quad(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\beta,\mathbf{w}_1) = \mathbf{T})$	11 Σ I
13	$\quad\quad\quad\neg(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{T})$	6 Reiteration
14	$\quad\quad\mathbf{v_I}(\beta,\mathbf{w}_1) = \mathbf{F})$	8-13 $\neg$ E
15	$\quad\mathbf{Rww}_1 \rightarrow \mathbf{v_I}(\beta,\mathbf{w}_1) = \mathbf{F}$	7-14 $\rightarrow$ I
16	$\quad(\Pi\mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{F})$	15 Π I
17	$\quad\mathbf{v_I}(\Diamond\beta,\mathbf{w}) = \mathbf{F}$	16 **SR**-$\Diamond$
18	$\quad\mathbf{v_I}(\Diamond\beta,\mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\Diamond\beta,\mathbf{w}) = \mathbf{F}$	17 $\bigvee$ I
19	$\mathbf{v_I}(\Diamond\beta,\mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\Diamond\beta,\mathbf{w}) = \mathbf{F}$	2 3-5 6-18 $\bigvee$ E

Case 7. α is $\Box\beta$. Given that Excluded Middle holds at the meta-logical level, there are exactly two kinds of interpretations **I**: At all accessible worlds $\mathbf{w}_i$, $\mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{T}$; it is not the case that at all accessible worlds $\mathbf{w}_i$, $\mathbf{v_I}(\beta,\mathbf{w}_i) = \mathbf{T}$. It will be shown that on either kind of interpretation, a sentence of the form $\Box\beta$ has the value **T** or the value **F** at an arbitrary world, in which case it has the value **T** or **F** at all worlds on all interpretations.

1	$(\Pi \mathbf{w})(\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}) \bigvee \mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{F})$	Inductive Hypothesis
2	$(\Pi \mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T}) \bigvee \neg(\Pi \mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T})$	Logic
3	$(\Pi \mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T})$	Assumption
4	$\mathbf{v_I}(\Box\beta, \mathbf{w}) = \mathbf{T}$	3 **SR**-□
5	$\mathbf{v_I}(\Box\beta, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\Box\beta, \mathbf{w}) = \mathbf{F}$	4 $\bigvee$ I
6	$\neg(\Pi \mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T})$	Assumption
7	$(\Sigma \mathbf{w}_i)\neg(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T})$	6 Quantifier-Negation
8	$(\Sigma \mathbf{w}_i)(\mathbf{Rww}_i \wedge \neg\mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T})$	7 Sentential Logic
9	$\mathbf{Rww}_1 \wedge \neg\mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{T}$	Assumption
10	$\neg\mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{T}$	9 $\bigwedge$ E
11	$\mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{T} \bigvee \mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{F}$	1 Π E
12	$\mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{F}$	10 11 Disjunctive Syllogism
13	$\mathbf{Rww}_1$	9 $\bigwedge$ E
14	$\mathbf{Rww}_1 \bigwedge \mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{F}$	12 13 $\bigwedge$ I
15	$(\Sigma \mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{F})$	14 Σ I
16	$(\Sigma \mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{F})$	8 9-15 Σ E
17	$\mathbf{v_I}(\Box\beta, \mathbf{w}) = \mathbf{F}$	16 **SR**-□
18	$\mathbf{v_I}(\Box\beta, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\Box\beta, \mathbf{w}) = \mathbf{F}$	17 $\bigvee$ I
19	$\mathbf{v_I}(\Box\beta, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\Box\beta, \mathbf{w}) = \mathbf{F}$	2 3-5 6-18 $\bigvee$ E

Case 8. α is $\beta \strictif \gamma$.

Proofs of cases 3-5 and 8 are left as an exercise for the reader.

Having shown that Modal Bivalence holds in the Basis Step and, given the Inductive Hypothesis, for all complex sentences of *MSL*, we conclude that it holds for all sentences of *MSL*.

1.2.2 Modal Truth-Functionality

As with Modal Bivalence, we must modify the definition of Truth-Functionality to accommodate the relativization of truth-values to possible worlds. The new definition is that on every interpretation **I**, and at every world **w** in **I**, the truth-value of a given sentence α at **w** is not both **T** and **F**. This property will be called 'Modal Truth-Functionality,' which will be abbreviated as '**MTF**.'

MTF. $\neg(\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T} \bigwedge \mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{F})$

Most of the inductive proof of Modal Truth-Functionality can be modified straightforwardly as in the above partial proof of Modal Bivalence. The Basis Step is fulfilled by the modal versions of **SR**-TVA and **SR**-⊥. The Inductive Hypothesis is made for all sentences α with fewer than *n* operators. Sentences with a non-modal main operator are not affected by values at other worlds. So the only cases of interest are those involving sentences whose main operator is modal. We will here only prove one case, where α is $\Diamond\beta$.

1	$(\Pi \mathbf{w}_i)\neg(\mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T} \wedge \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{F})$	Inductive Hypothesis
2	$\quad \mathbf{v_I}(\Diamond\beta, \mathbf{w}_i) = \mathbf{T} \wedge \mathbf{v_I}(\Diamond\beta, \mathbf{w}_i) = \mathbf{F}$	Assumption
3	$\quad \mathbf{v_I}(\Diamond\beta, \mathbf{w}_i) = \mathbf{T}$	2 $\wedge$ E
4	$\quad \mathbf{v_I}(\Diamond\beta, \mathbf{w}_i) = \mathbf{F}$	2 $\wedge$ E
5	$\quad (\Sigma \mathbf{w}_i)(\mathbf{Rww}_i \wedge \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T})$	3 **SR**-$\Diamond$
6	$\quad (\Pi \mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{F})$	4 **SR**-$\Diamond$
7	$\quad\quad \mathbf{Rww}_1 \wedge \mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{T}$	Assumption
8	$\quad\quad \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T}$	7 $\wedge$ E
9	$\quad\quad \mathbf{Rww}_1$	7 $\wedge$ E
10	$\quad\quad \mathbf{Rww}_1 \rightarrow \mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{F}$	6 Π E
11	$\quad\quad \mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{F}$	9 10 $\rightarrow$ E
12	$\quad\quad \mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T} \wedge \mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{F}$	8 11 $\wedge$ I
13	$\quad\quad \neg(\mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{T} \wedge \mathbf{v_I}(\beta, \mathbf{w}_1) = \mathbf{F})$	1 Π E
14	$\quad\quad Falsum$	12 13 *Falsum* I
15	$\quad Falsum$	5 7-14 Σ E
16	$\neg(\mathbf{v_I}(\Diamond\beta, \mathbf{w}_i) = \mathbf{T} \wedge \mathbf{v_I}(\Diamond\beta, \mathbf{w}_i) = \mathbf{F})$	2-15 $\neg$ I-*Falsum*

1.2.3 Semantical Entailment in *KI*

Recall that a sentence α is said to be semantically entailed in a frame **Fr** by a set of sentences $\{\gamma_1, \ldots, \gamma_n\}$, if and only if at every world **w** on every interpretation **I** based on **Fr**, if $\mathbf{v_I}(\gamma_1, \mathbf{w}) = \mathbf{T}$, and ..., and $\mathbf{v_I}(\gamma_n, \mathbf{w}) = \mathbf{T}$, then $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$. Now we can say that $\{\gamma_1, \ldots, \gamma_n\}$ *KI-entails* α, $\{\gamma_1, \ldots, \gamma_n\} \vDash_{KI} \alpha$, just in case entailment in a frame holds in every *KI*-frame, which is to say in every frame.

1.2.4 Semantical Equivalence in *KI*

Two sentences α and β are *KI-equivalent* when they are have the same truth-value at all worlds on all interpretations based on any *KI*-frame, i.e., on any frame. Thus, two sentences are *KI*-equivalent if and only if for all frames, they are equivalent in those frames.

1.2.5 Validity in *KI*

A sentence is *K-valid*, $\vDash_{KI} \alpha$, when it has the value **T** at all worlds on all interpretations based on any *KI*-frame, and so on any frame. So, a *K*-valid sentence is valid in all frames.

1.2.6 Semantical Consistency in *KI*

In order to get an appropriate notion of semantical consistency, we must again treat $\mathbf{v_I}$ as a two-place function. We shall say that a set of sentences Γ of *MSL* is semantically consistent in *KI* if and only if there is an interpretation based on a *KI*-frame such that there is a world at which all the sentences in Γ have the value **T**.

Semantical Consistency in *KI*

Γ is semantically consistent in *KI* if and only if there is an interpretation **I** based on a *KI* frame **Fr** in which there is a world **w** in the set of worlds **W** such that for all γ_i in Γ, $\mathbf{v_I}(\gamma_1, \mathbf{w})$=**T**, and . . . and $\mathbf{v_I}(\gamma_n, \mathbf{w})$=**T**; Γ is semantically inconsistent in *KI* if and only if there is no interpretation **I** based on a frame **Fr** in which there is a world **w** in the set of worlds **W** such that for all γ_i in Γ, $\mathbf{v_I}(\gamma_1, \mathbf{w})$=**T**, and . . . and $\mathbf{v_I}(\gamma_n, \mathbf{w})$=**T**.

We can prove a result for *KI* analogous to that for *SI*:

$\Gamma \vDash_{KI} \alpha$ if and only if $\Gamma \cup \sim\alpha$ is s-inconsistent in *KI*.

$\Gamma \vDash_{KI} \alpha$ if and only if for any interpretation **I** based on a *KI*-frame **Fr**, any **w** in **W** in **Fr**, and any γ_i in Γ, if $\mathbf{v_I}(\gamma_1, \mathbf{w}) = \mathbf{T}$, and . . ., and $\mathbf{v_I}(\gamma_n, \mathbf{w}) = \mathbf{T}$, then $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$. By Modal Truth-Functionality, this holds if and only if, if $\mathbf{v_I}(\gamma_i, \mathbf{w}) = \mathbf{T}$, then $\neg\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{F}$. By **SR-**$\sim$, this holds if and only if, if $\mathbf{v_I}(\gamma_i, \mathbf{w}) = \mathbf{T}$, then $\neg\mathbf{v_I}(\sim\alpha, \mathbf{w}) = \mathbf{T}$. Finally, this result holds if and only if there is no interpretation **I** based on a *KI*-frame **Fr** whose set of worlds **W** contains a world **w** such that all the members γ_i of Γ have the value **T** at **w** and α has the value **T** at **w**, i.e. $\Gamma \cup \sim\alpha$ is s-inconsistent in *KI*.

1.3 An Illustration

To illustrate the workings of the semantical system *KI*, we shall prove that

$\{\Box(A \supset B), \Box A\} \vDash_{KI} \Box B$.

Let **Fr** be an arbitrary *KI*-frame, **w** be a world in **W** in **Fr**, and **I** be an arbitrary interpretation based on **Fr**. Suppose $\mathbf{v_I}(\Box(A \supset B), \mathbf{w}) = \mathbf{T}$ and $\mathbf{v_I}(\Box A, \mathbf{w}) = \mathbf{T}$. It follows by two applications of **SR**-$\Box$ that for all worlds $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, $\mathbf{v_I}(A \supset B, \mathbf{w}_i) = \mathbf{T}$ and $\mathbf{v_I}(A, \mathbf{w}_i) = \mathbf{T}$. Suppose $\mathbf{Rww}_i$. Then $\mathbf{v_I}(A \supset B, \mathbf{w}_i) = \mathbf{T}$ and $\mathbf{v_I}(A, \mathbf{w}_i) = \mathbf{T}$. By **SR**-$\supset$, $\mathbf{v_I}(A, \mathbf{w}_i) = \mathbf{F}$ or $\mathbf{v_I}(B, \mathbf{w}_i) = \mathbf{T}$. By Modal Truth-Functionality, it is not the case that $\mathbf{v_I}(A, \mathbf{w}_i) = \mathbf{F}$, in which case $\mathbf{v_I}(B, \mathbf{w}_i) = \mathbf{T}$. Thus for arbitrary $\mathbf{w}_i$, if $\mathbf{Rww}_i$, then $\mathbf{v_I}(B, \mathbf{w}_i) = \mathbf{T}$. It follows by **SR**-$\Box$ that $\mathbf{v_I}(\Box B, \mathbf{w}) = \mathbf{T}$. Since the choice of **w** and **I** were arbitrary, $\{\Box(A \supset B), \Box A\} \vDash_{\mathbf{Fr}} \Box B$. And since the choice of **Fr** was arbitrary, $\{\Box(A \supset B), \Box A\} \vDash_{KI} \Box B$.

Semantical proof that $\{\Box(A \supset B), \Box A\} \vDash_{KI} \Box B$

1	$\mathbf{v_I}(\Box(A \supset B), \mathbf{w}) = \mathbf{T}$	Assumption
2	$\mathbf{v_I}(\Box A, \mathbf{w}) = \mathbf{T}$	Assumption
3	$(\Pi \mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(A \supset B, \mathbf{w}_i) = \mathbf{T})$	1 **SR**-□
4	$(\Pi \mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(A, \mathbf{w}_i) = \mathbf{T})$	2 **SR**-□
5	$\mathbf{Rww}_1 \rightarrow \mathbf{v_I}(A \supset B, \mathbf{w}_1) = \mathbf{T}$	3 Π E
6	$\mathbf{Rww}_1 \rightarrow \mathbf{v_I}(A, \mathbf{w}_1) = \mathbf{T}$	4 Π E
7	$\mathbf{Rww}_1$	Assumption
8	$\mathbf{v_I}(A \supset B, \mathbf{w}_1) = \mathbf{T}$	5 7 → E
9	$\mathbf{v_I}(A, \mathbf{w}_1) = \mathbf{T}$	6 7 → E
10	$\mathbf{v_I}(A, \mathbf{w}_1) = \mathbf{F} \bigvee \mathbf{v_I}(B, \mathbf{w}) = \mathbf{T}$	8 **SR**-⊃
11	$\neg \mathbf{v_I}(A, \mathbf{w}_1) = \mathbf{F}$	9 Modal Truth-Functionality
12	$\mathbf{v_I}(B, \mathbf{w}_1) = \mathbf{T}$	10 11 Disjunctive Syllogism
13	$\mathbf{Rww}_i \rightarrow \mathbf{v_I}(B, \mathbf{w}_1) = \mathbf{T}$	7-12 → I
14	$(\Pi \mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(B, \mathbf{w}_i) = \mathbf{T})$	13 Π I
15	$\mathbf{v_I}(\Box B, \mathbf{w}) = \mathbf{T}$	14 **SR**-□

1.4 Closure in *KI*

In the previous example, it was proved that '□Q' is semantically entailed in *KI* by the sentences '□P' and '□(P ⊃ Q).' Notice that it is also the case that 'Q' is semantically entailed by 'P' and 'P ⊃ Q' in the semantical system *SI* for Sentential Logic: $\{P, P \supset Q\} \vDash_{SI} Q$. These two results suggest that in the semantical system *KI*, if $\{P, P \supset Q\} \vDash_{SI} Q$, then $\{\Box P, \Box(P \supset Q)\} \vDash_{KI} \Box Q$ The reason is that the semantical rules for the non-modal operators function normally in any accessible world.

A more general result for *KI* is that *KI*-entailment is preserved when the sentences (including modal sentences) embodying a *KI*-entailment are made into necessity-sentences. The '□' operator usually described as being is *closed* under the relation of *KI*-entailment.

Closure of the '□' under *KI*-Entailment

If $\{\gamma_1, \ldots, \gamma_n\} \vDash_{KI} \alpha$, then $\{\Box\gamma_1, \ldots, \Box\gamma_n\} \vDash_{KI} \Box\alpha$

Proof. Suppose $\{\gamma_1, \ldots, \gamma_n\} \vDash_{KI} \alpha$. Then for any *KI*-frame **Fr**, any **I** based on **Fr**, and any **w** in **W**, if $\mathbf{v_I}(\gamma_1, \mathbf{w})=\mathbf{T}$ and, …, and $\mathbf{v_I}(\gamma_n, \mathbf{w})=\mathbf{T}$, then $\mathbf{v_I}(\alpha, \mathbf{w})=\mathbf{T}$. Suppose further that $\mathbf{v}_I(\Box\gamma_1, \mathbf{w})=\mathbf{T}$, and, …, and $\mathbf{v_I}(\Box\gamma_n, \mathbf{w})=\mathbf{T}$. Then for each γ_i, $\mathbf{v_I}(\gamma_i,\mathbf{w}_i)=\mathbf{T}$ for each world $\mathbf{w}_i$ accessible to **w**. From the original supposition, it follows that if $\mathbf{v_I}(\gamma_1, \mathbf{w}_i)=\mathbf{T}$ and, …, and $\mathbf{v_I}(\gamma_n, \mathbf{w}_i)=\mathbf{T}$, then $\mathbf{v_I}(\alpha, \mathbf{w}_i)=\mathbf{T}$. So $\mathbf{v_I}(\alpha,\mathbf{w}_i)=\mathbf{T}$ for each $\mathbf{w}_i$ accessible to **w**. Therefore, by **SR**-□, $\mathbf{v_I}(\Box\alpha,\mathbf{w})=\mathbf{T}$. So, if $\mathbf{v}_I(\Box\gamma_1, \mathbf{w})=\mathbf{T}$, and …, and $\mathbf{v_I}(\Box\gamma_n, \mathbf{w})=\mathbf{T}$, then $\mathbf{v_I}(\Box\alpha,\mathbf{w})=\mathbf{T}$. Since the result is obtained with arbitrary **I** and **w**, it follows that $\{\Box\gamma_1, \ldots, \Box\gamma_n\} \vDash_{\mathbf{Fr}} \Box\alpha$. And since the *KI*-frame **Fr** was arbitrary, we have it that $\{\Box\gamma_1, \ldots, \Box\gamma_n\} \vDash_{KI} \Box\alpha$, which was to be demonstrated.

KI-validity is a special case of *KI*-entailment, where $\{\gamma_1, \ldots, \gamma_n\}$ is the empty set: $\emptyset \vDash_{KI} \alpha$. So we can obtain a corollary of Closure in the case of validity. We will call this special case of closure *necessitation*.

Necessitation of *KI*-valid Sentences

If $\vDash_{KI} \alpha$, then $\vDash_{KI} \Box\alpha$

Proof. Suppose $\vDash_{KI} \alpha$. Then $\mathbf{v_I}(\alpha, \mathbf{w})$=**T** for all worlds **w** in all *KI*-interpretations **I** based on any *KI*-frame **Fr**. Suppose $\mathbf{Rww}_i$. Then $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T**. So if $\mathbf{Rww}_i$, then $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T**. Then by **SR**-□, $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**T**. Since this holds for any **I** and **w**, $\vDash_{\mathbf{Fr}} \Box\alpha$. And since it holds for any frame **Fr**, $\vDash_{KI} \Box\alpha$.

2 The Derivational System *KD*

The derivational system *KD* consists of the *SD* rules together with the special rules for modal sentences and the restriction that reiteration not occur across strict scope lines.

2.1 Specification of *KD*

The special modal rules include Strict Reiteration for '□,' '∼◊,' '◊,' and '⥽.' Other modal rules are the Introduction rules for '□,' '∼◊,' and '⥽,' as well as the rule of ◊ Elimination.[2] A sentence α is *KD-derivable* from a set of sentences $\{\gamma_1, \ldots, \gamma_n\}$, or more compactly, $\{\gamma_1, \ldots, \gamma_n\} \vdash_{KD} \alpha$, just in case α is the last step in a derivation by *KD* rules from the members of $\{\gamma_1, \ldots, \gamma_n\}$ as assumptions and itself lies in the scope of no other assumptions or strict scope lines. A sentence α is a *KD-theorem*, $\vdash_{KD} \alpha$, if and only if it is derivable by these rules from no undischarged assumptions and is not to the right of any strict scope line.

The definition of derivational consistency for *SD* can be carried over directly to *KD*. A set of sentences Γ of *MSL* is derivationally consistent (d-consistent) in *KD*, if and only if it is not the case that there is a sentence α of *MSL* such that $\Gamma \vdash_{KD} \alpha$ and $\Gamma \vdash_{KD} \sim\alpha$. Any set of sentences which is not d-consistent is *d-inconsistent* in *KD*. This definition is modified only by reference to the modal language and derivational system. The reason is that none of the rules for modal operators change the definition of a derivation of α from Γ. All restricted scope lines must be discharged for there to be a derivation. We may then carry over our result from *SD*:

$\Gamma \vdash_{KD} \alpha$ if and only if $\Gamma \cup \sim\alpha$ is d-inconsistent in *KD*.

The proof of this meta-theorem is the same as for *KD*, since restricted scope lines are not relevant.

2.2 Closure in *KD*

Corresponding to closure under semantic entailment is closure under derivability.

Closure of the '□' under *KD*-Derivability

$\{\gamma_1, \ldots, \gamma_n\} \vdash_{KD} \alpha$, then $\{\Box\gamma_1, \ldots, \Box\gamma_n\} \vdash_{KD} \Box\alpha$

We can demonstrate closure under *KD*-derivability by providing a schema representing a style of derivation.

[2]Stronger derivational systems are generated by adding to the modal rules for *KD*.

Schema for Closure Derivations

$\Box\gamma_1$	Assumption
...	
$\Box\gamma_n$	Assumption
$\Box$ | γ_1	SR-$\Box$
| ...	
| γ_n	SR-$\Box$
| ...	
| α	Derivable from $\{\gamma_1 \ldots \gamma_n\}$
$\Box\alpha$	$\Box$ I

We first assume each of $\Box\gamma_1, \ldots, \Box\gamma_n$. Then we introduce a restricted scope line and strictly reiterate each of the assumptions. Given that α is derivable from $\{\gamma_1, \ldots, \gamma_n\}$ with no undischarged assumptions, this derivation can take place entirely within the restricted scope line. And if this is so, then we can end the restricted scope line with the use of $\Box$ Introduction. Therefore, the derivation may be ended and $\Box\alpha$ is derivable from $\{\Box\gamma_1, \ldots, \Box\gamma_n\}$, which was to be proved.

Corresponding to "necessitation" in the semantical system is a meta-theorem of the derivational system:

Necessitation of *KD*-Theorems

If $\vdash_{KD} \alpha$, then $\vdash_{KD} \Box\alpha$

Derivational systems (and axiom systems) which have the property of necessitation are called *normal* systems.[3]

We can easily give a derivation schema showing that the result holds for *KI*-derivability.

Schema for Necessitation Derivations

$\Box$ | ...	
| α	Derivable from no undischarged assumptions
$\Box\alpha$	$\Box$ I

If it is assumed that α is derivable with no undischarged assumptions, then the derivation can take place entirely within a restricted scope line. And if this is so, then we can end the restricted scope line with the use of $\Box$ Introduction. This can be done within the leftmost restricted scope line, and so the derivation may be ended and $\Box\alpha$ is a theorem of *KD*.

We assert without proof that the derivational rules of *KD* are sound and complete with respect to the semantical system *KI*.

[3]The early axiom system *S3* of C.I. Lewis is not a normal system. Semantics for this and his later systems *S1* and *S2* are not even indirectly truth-functional. Semantically, frames contain "non-normal" worlds at which some non-atomic sentences are assigned truth-values by the valuation function directly, rather than on the basis of the truth-values of component sentences at accessible worlds. Specifically, $\Box\alpha$ is always false and $\Diamond\alpha$ always true.

Soundness and Completeness (for *KD*)

$\{\gamma_1, \ldots, \gamma_n\} \vdash_{KD} \alpha$ if and only if $\{\gamma_1, \ldots, \gamma_n\} \vDash_{KI} \alpha$.

We shall frequently use the derivational system to show semantic entailment and validity. It is easier to use than the semantical system, and derivations can be converted automatically into semantical arguments. To show non-entailment, non-validity and other properties that do not require the kind of general reasoning we have been using to establish entailment and validity, we shall construct specific interpretations or describe specific types of interpretations. These interpretations cannot be converted into derivations, as these properties cannot be established by derivation.

3 Axiomatic Equivalents of *KD*

Most expositions of modal logic present modal systems as axiom systems rather than derivability systems.[4] What follows is a very brief introduction to the axiomatic approach, which may be of value to the reader in understanding other work on modal logic.

The goal of the logicians who originally produced axiom systems of modal logic was generate a set of theorems that correspond to true sentences about the modalities in question. The resulting system is then said to be a *theory* of the modalities on the analogy of a theory of sets or a theory of numbers. The main approach in this text is to focus on *inferences* made with modal premises and/or modal conclusions. It will be seen shortly that this can be done within the scope of an axiom system, as it is possible to define a derivability relation for the axiom system like that for *KD*.

There are many axiomatic equivalents of *KD* and other derivational systems of modal logic. We can call each one of them *K*, in the sense that they all have the same set of theorems. In what follows, we will sketch the most common axiom system for *K*.

3.1 An Axiomatization of *K*

In an earlier module, an axiom system for non-modal Sentential Logic, system *SLA*, was presented. There were three axiom schemata and one non-modal rule of inference. The system *K* is an extension of the system *SLA*. It preserves the axiom schemata and inference rule of *SLA* while adding one non-modal axiom schema and one non-modal rule of inference of *SLA*. The formulation of *K* given here takes the '□' operator as primitive.

Axiomatization of *K*

K-1	$\vdash_K \alpha \supset (\beta \supset \alpha)$,
K-2	$\vdash_K (\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$,
K-3	$\vdash_K ({\sim}\beta \supset {\sim}\alpha) \supset (({\sim}\beta \supset \alpha) \supset \beta)$,
K	$\vdash_K \Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$,
Modus Ponens	From $\vdash_K \alpha$ and $\vdash_K \alpha \supset \beta$, infer $\vdash_K \beta$.
Necessitation (Nec)	From $\vdash_K \alpha$, infer $\vdash_K \Box\alpha$.

The following are schemata for two significant theorems of *K*, the proof of which requires the use of **Necessitation**:

$\vdash_K \Box(\alpha \wedge \beta) \equiv (\Box\alpha \wedge \Box\beta)$,

[4]Notable exceptions are Frederick Fitch, *Symbolic Logic*, Kenneth Konyndyk, *Introductory Modal Logic* and Daniel Bonevac, *Deduction*.

$\vdash_K (\Box\alpha \vee \Box\beta) \supset \Box(\alpha \vee \beta)$.

It has been proved elsewhere, and not in this text, that *K* is sound and complete relative to the semantical system *KI*. We will assume these results here . To bring out the analogy between *K* and the derivational system *KD*, we shall give a Fitch-style meta-logical derivation of the validity of the axiom schema **K**.

Before doing so, we shall first make note of a derived semantical rule for '⊃' in *KI*:

SR-⊃D $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w}) = \mathbf{T}$ if and only if, if $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$, then $\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}$.

Semantical proof that $(\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T} \rightarrow \mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}) \rightarrow \mathbf{v_I}(\alpha \supset \beta, \mathbf{w}) = \mathbf{T}$:

1	$\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{F}$	Modal Bivalence
2	$\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T} \bigvee \mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{F}$	Modal Bivalence
3	$\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T} \rightarrow \mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}$	Assumption
4	$\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}$	Assumption
5	$\mathbf{v_I}(\alpha \supset \beta, \mathbf{w}) = \mathbf{T}$	4 **SR**-⊃
6	$\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{F}$	Assumption
7	$\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$	Assumption
8	$\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}$	3 7→ E
9	$\mathbf{v_I}(\alpha \supset \beta, \mathbf{w}) = \mathbf{T}$	8 **SR**-⊃
10	$\mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{F}$	Assumption
11	$\mathbf{v_I}(\alpha \supset \beta, \mathbf{w}) = \mathbf{T}$	10 **SR**-⊃
12	$\mathbf{v_I}(\alpha \supset \beta, \mathbf{w}) = \mathbf{T}$	1 7-9 10-11 ⋁ E
13	$\mathbf{v_I}(\alpha \supset \beta, \mathbf{w}) = \mathbf{T}$	2 4-5 6-12 ⋁ E
14	$(\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T} \rightarrow \mathbf{v}(\beta, \mathbf{w}) = \mathbf{T}) \rightarrow \mathbf{v}(\alpha \supset \beta, \mathbf{w}) = \mathbf{T}$	3-13 → I

Exercise. Prove the converse, that is, if $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w}) = \mathbf{T}$, then if $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$, then $\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}$

We now show that **K**, the non-modal axiom schema for *K*, has only *KI*-valid instances.

Semantical proof of: $\vDash_{KI} \Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$

1	$\mid \mathbf{v}(\Box(\alpha \supset \beta), \mathbf{w}) = \mathbf{T}$	Assumption
2	$\mid \mid \mathbf{v}(\Box\alpha, \mathbf{w}) = \mathbf{T}$	Assumption
3	$\mid \mid \mid \mathbf{Rww_1}$	Assumption
4	$\mid \mid \mid \mathbf{v}(\alpha \supset \beta, \mathbf{w_1}) = \mathbf{T}$	1 3 **SR**-□C
5	$\mid \mid \mid \mathbf{v}(\alpha, \mathbf{w_1}) = \mathbf{T}$	2 3 **SR**-□C
6	$\mid \mid \mid \mathbf{v}(\beta, \mathbf{w_1}) = \mathbf{T}$	4 5 **SR**-⊃D
7	$\mid \mid \mathbf{Rww_1} \rightarrow \mathbf{v}(\beta, \mathbf{w_1}) = \mathbf{T}$	3-6 → I
8	$\mid \mid \mathbf{v}(\Box\beta, \mathbf{w}) = \mathbf{T}$	7 **SR**-□
9	$\mid \mathbf{v}(\Box\alpha, \mathbf{w}) = \mathbf{T} \rightarrow \mathbf{v}(\Box\beta, \mathbf{w}) = \mathbf{T}$	2-8 → I
10	$\mid \mathbf{v}(\Box\alpha \supset \Box\beta, \mathbf{w}) = \mathbf{T}$	9 **SR**⊃D
11	$\mathbf{v}(\Box(\alpha \supset \beta), \mathbf{w}) = \mathbf{T} \rightarrow \mathbf{v}(\Box\alpha \supset \Box\beta, \mathbf{w}) = \mathbf{T}$	1-10 → I
12	$\mathbf{v}(\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta), \mathbf{w}) = \mathbf{T}$	11 **SR**⊃D

3.2 *K* and *KD*

The axiom system *K* generates all the same theorems as does the derivational system *KD*. Here we shall sketch a proof of half of this result, that all theorems of *K* are derivable in *KD*. We assume that the non-modal components of *K* (*SA*) and of *KD* (*SD*) have this property. It first must be shown that all the modal axioms of *K* are derivable in *KD*. We here derive one axiom and take it to be apparent that any other axiom could be derived in exactly the same way.

Derivation of an Instance of the Modal *K* Axiom

1	$\mid \Box(A \supset B)$	Assumption
2	$\mid \mid \Box A$	Assumption
3	$\mid \mid \Box(A \supset B)$	1 Reiteration
4	$\mid \mid \Box \mid A \supset B$	3 SR-□
5	$\mid \mid \ \ \mid A$	2 SR-□
6	$\mid \mid \ \ \mid B$	4 5 ⊃ E
7	$\mid \mid \Box B$	2 4-6 □ I
8	$\mid \Box A \supset \Box B$	2-7 ⊃ I
9	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	1-8 ⊃ I

We have already shown that use of the rule of inference, Necessitation, can be simulated using derivations. So we can use the derivational system *KD* to prove all the theorems of *K*. *KD* is said to *contain* *K* in this sense. To prove the equivalence, relative to the set of theorems, of *KD* and *K*, one would have to show that all the theorems of *KD* are theorems of *K*. This is not easy to do, and we shall not undertake such a proof here.

One way to achieve the desired result would be to take a detour through the semantics. If it can be shown that all the theorems of *KD* are valid in the semantical system *KI* (soundness of *KD*), and that all *KI*-valid sentences are theorems of *K* (completeness of *K*), it would follow that all theorems of *KD* are theorems of *K*.

3.3 Derivability in *K*

Another way to express the relation between *K* and *KD* is by defining a derivability relation in *K*. We want to be able to give meaning to the notation that $\{\gamma_1, \ldots, \gamma_n\} \vdash_K \alpha$. To do so, we partially define a *derivation* in *K* of α from $\{\gamma_1, \ldots, \gamma_n\}$ as a finitely long, non-empty, list of sentences of *MSL* such that each sentence in the list is either:

1. A member of the set $\{\gamma_1, \ldots, \gamma_n\}$, or
2. An axiom of *K*, or
3. An immediate consequence by **Modus Ponens** applied to two sentences above it in the list as if they were theorems.

Here is an example of a *K*-derivation that proves that $\{\Box A, \Box(A \supset B)\} \vdash_K \Box B$.

1	$\Box A$	Member of the set $\{\Box A, \Box(A \supset B)\}$
2	$\Box(A \supset B)$	Member of the set $\{\Box A, \Box(A \supset B)\}$
3	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	**K**
4	$\Box A \supset \Box B$	2 3 **Modus Ponens**
5	$\Box B$	1 4 **Modus Ponens**

The reason the definition of a derivation in *K* is only partial is that there is a problem with the *K* rule of inference, **Necessitation**. This rule allows us to infer from the fact that α is a theorem of *K* to the conclusion that $\Box\alpha$ is a theorem of *K*. It cannot, however, be applied to sentences that are merely members of Γ. If it could, we would have the following derivation of '$\Box A$' from $\{A\}$, which would collapse the distinction between modal and non-modal sentential logic.

1	A	Member of the set $\{A\}$
2	$\Box A$	1 **Nec**

To circumvent this problem, we will not allow **Nec** to be applied in derivations. Instead, we will rely on the generalized form of **Nec**, which in *KD* is a property we called 'Closure.' The rule will be as follows:

Closure. If there is a derivation of α from the set $\{\gamma_1, \ldots, \gamma_n\}$, then there is a derivation of $\Box\alpha$ from $\{\Box\gamma_1, \ldots, \Box\gamma_n\}$.[5]

Here is an example of a *K*-derivation to which the rule can be applied. First, we derive '$A \supset B$' from $\{B\}$.

1	B	Member of the set $\{B\}$
2	$B \supset (A \supset B)$	**K**-1
3	$A \supset B$	1 2 **Modus Ponens**

The derivation generated by **Closure** will be the following:

1	$\Box B$	Member of the set $\{\Box B\}$
2	$\Box(A \supset B)$	1, $\{B\} \vdash_K A \supset B$, **Closure**

[5]See Hughes and Cresswell, *A New Introduction to Modal Logic*, p. 214.

Thus, we can assert that $\{\Box B\} \vdash_K \Box(A \supset B)$. The use of **Closure** here exactly parallels the use of Strict Reiteration for '□' and □ Introduction.

Derivation of $\Box(A \supset B)$ from $\{\Box B\}$

1	$\Box B$	Assumption
2	□ | B	1 SR-□
3	| | A	Assumption
4	| | B	2 Reiteration
5	| $A \supset B$	3-4 ⊃ I
6	$\Box(A \supset B)$	1 3-5 □ I

The fact that '$A \supset B$' can be derived from $\{B\}$ means that the derivation can take place to the right of a restricted scope line.

The completeness and soundness of K relative to KI yields the result that: $\{\gamma_1, \ldots, \gamma_n\} \vdash_K \alpha$ if and only if $\{\gamma_1, \ldots, \gamma_n\} \vDash_{KI} \alpha$. A more general result, which will not be proved here, is that $\{\gamma_1, \ldots, \gamma_n\} \vdash_K \alpha$ if and only if $\{\gamma_1, \ldots, \gamma_n\} \vdash_{KD} \alpha$. The consequence-relation of derivability gives perfectly parallel results in the three systems, and in this sense the three can be said to be equivalent.

4 Modal Operators in *K* and its Equivalents

Thus far *K* as been presented in terms of the '□' operator. In what follows, the other modal operators of *MSL* will be discussed in more detail in the context of *K* and its equivalents.

4.1 Possibility

As has been noted, necessity-sentences and possibility-sentences can be defined in terms of each other. If a necessity-sentence is defined in terms of a possibility-sentence, or *vice-versa*, then the defined sentence is just a notational variant of the original sentence. In that case, we can say that these sentences are *definitionally equivalent*.

In the semantics, two sentences are equivalent just in case they have the same truth values at all worlds on all interpretations. In the derivational system, two sentences are equivalent just in case each is derivable from the other as an assumption, with no other undischarged assumptions.

In the semantical and derivational systems as presented in the last module, sentences whose main operator is '□' or '◊' are governed by their own set of rules and are not defined in terms of each other. Rather than providing definitions, we prove equivalences in *KI* and *KD* which parallel the definitions that might have been given. These and two other equivalences are commonly known as *Duality*.

Duality

$\Box\alpha$ is semantically and derivationally equivalent to $\sim\Diamond\sim\alpha$

$\Diamond\alpha$ is semantically and derivationally equivalent to $\sim\Box\sim\alpha$

$\Box\sim\alpha$ is semantically and derivationally equivalent to $\sim\Diamond\alpha$

$\Diamond\sim\alpha$ is semantically and derivationally equivalent to $\sim\Box\alpha$

For the semantical system, we use the truth-definitions for the '□' and the '◊' to prove the equivalence. For the derivational system, the rules of inference for the two operators are used to complete the proof. We will prove the first of the four semantic equivalences and leave the proof of the other three as an exercise.

On an arbitrary interpretation **I** and arbitrary world **w** in **I** in an arbitrary *KI*-frame,
$\mathbf{v_I}(\Box\alpha, \mathbf{w})=\mathbf{T}$ iff
for all $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, $\mathbf{v_I}(\alpha, \mathbf{w}_i)=\mathbf{T}$ iff
for all $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, $\mathbf{v_I}(\sim\alpha, \mathbf{w}_i)=\mathbf{F}$ iff
for all $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, it is not the case that $\mathbf{v_I}(\sim\alpha, \mathbf{w}_i)=\mathbf{T}$ iff
$\mathbf{v_I}(\Diamond\sim\alpha, \mathbf{w})=\mathbf{F}$ iff
$\mathbf{v_I}(\sim\Diamond\sim\alpha, \mathbf{w})=\mathbf{T}$.

Since the choice of interpretations and worlds is arbitrary, $\Box\alpha$ and $\sim\Diamond\sim\alpha$ have the same truth-value on all interpretations, and so they are semantically equivalent in an arbitrary frame. Since the choice of frames is arbitrary, the sentences instantiating the schemata are equivalent in all frames *KI*-equivalent.

Exercise Show that the other three Duality semantical equivalences hold.

To show that the $\Box\alpha$ and $\sim\Diamond\sim\alpha$ are inter-derivable, we must give two separate derivations. We will use the rule-sets for the '□' and for '∼◊,' since the rules for '◊' appear to be incomplete.

Proof that: $\{\Box\alpha\} \vdash_{KD} \sim\Diamond\sim\alpha$

1	$\Box\alpha$	Assumption
2	□ | α	1 SR-□
3	| | $\sim\alpha$	Assumption
4	| | α	2 Reiteration
5	| $\sim\sim\alpha$	3-4 ∼ I
6	$\sim\Diamond\sim\alpha$	1 2-5 ∼◊ I

Proof that: $\{\sim\Diamond\sim\alpha \vdash_{KD} \Box\alpha\}$

1	$\sim\Diamond\sim\alpha$	Assumption
2	□ | $\sim\sim\alpha$	1 SR-∼◊
3	| | $\sim\alpha$	Assumption
4	| | $\sim\sim\alpha$	2 Reiteration
5	| α	3-4 ∼ E
6	$\Box\alpha$	1 2-5 □ I

We prove half of the second derivational equivalence.

Proof that: $\{\Diamond\alpha\} \vdash_{KD} {\sim}\Box{\sim}\alpha$

1	$\Diamond\alpha$	Assumption
2	$\Box{\sim}\alpha$	Assumption
3	$\Diamond\alpha$	1 R
4	$\Box$ ${\sim}\alpha$	2 SR-$\Box$
5	${\sim}\Diamond\alpha$	2 4 ${\sim}\Diamond$ I
6	${\sim}\Box{\sim}\alpha$	2-5 ~ I

Exercise. Prove the other half of the second equivalence as well as the third and fourth equivalences.

The reader might suspect that some question has been begged in using the Impossibility rules to prove Duality, since those rules have their origin in the definitional versions of Duality given in an earlier module. But the definitions have not been presupposed to prove Duality in the derivational system. The Impossibility rules are motivated by Duality definitions but do not depend on the definitions in any formal way.

Given the Impossibility rules, *if* we define the '$\Diamond$' in terms of the '$\Box$', *then* the resulting derivations yield the definitional equivalents of the derivations using the '$\Box$' rules. What has been shown here is that given *both sets of rules*, we can demonstrate the equivalences given in the definitions without presupposing the definitions.

In axiomatic *K*, the following are schemata of theorems which use the '$\Diamond$' operator defined in terms of the '$\Box$' operator.

$\vdash_K \Diamond(\alpha \vee \beta) \equiv (\Diamond\alpha \vee \Diamond\beta)$

$\vdash_K \Diamond(\alpha \supset \beta) \equiv (\Box\alpha \supset \Diamond\beta)$

$\vdash_K \Diamond(\alpha \wedge \beta) \supset (\Diamond\alpha \wedge \Diamond\beta)$

$\vdash_K \Box(\alpha \vee \beta) \supset (\Box\alpha \vee \Diamond\beta)$

We will give proofs of one of these theorems and half of another in *KD*.

To prove: $\vdash_{KD} \Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$

1	$\Diamond(A \supset B)$	Assumption
2	$\Box A$	Assumption
3	$\Diamond(A \supset B)$	1 R
4	$\Box$ $A \supset B$	3 SR-$\Diamond$
5	A	2 SR-$\Box$
6	B	5 6 $\supset$ E
7	$\Diamond B$	3 4-7 $\Diamond$ E
8	$\Box A \supset \Diamond B$	2-8 $\supset$ I
9	$\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$	1-9 $\supset$ I

To prove: $\vdash_{KD} \Box(A \vee B) \supset (\Box A \vee \Diamond B)$

1	$\Box(A \vee B)$	Assumption
2	$\quad \sim(\Box A \vee \Diamond B)$	Assumption
3	$\quad \Box(A \vee B)$	1 R
4	$\quad\quad \Box A$	Assumption
5	$\quad\quad \Box A \vee \Diamond B$	4 $\vee$ I
6	$\quad\quad \sim(\Box A \vee \Diamond B)$	2 Reiteration
7	$\quad \sim\Box A$	3-6 $\sim$ I
8	$\quad\quad \Diamond B$	Assumption
9	$\quad\quad \Box A \vee \Diamond B$	8 $\vee$ I
10	$\quad\quad \sim(\Box A \vee \Diamond B)$	2 Reiteration
11	$\quad \sim\Diamond B$	8-10 $\sim$ I
12	$\quad \Box \quad A \vee B$	3 SR-$\Box$
13	$\quad\quad \sim B$	11 SR-$\sim\Diamond$
14	$\quad\quad\quad A$	Assumption
15	$\quad\quad\quad A$	14 Reiteration
16	$\quad\quad\quad B$	Assumption
17	$\quad\quad\quad\quad \sim A$	Assumption
18	$\quad\quad\quad\quad \sim B$	13 Reiteration
19	$\quad\quad\quad\quad B$	16 Reiteration
20	$\quad\quad\quad A$	17-19 $\sim$ E
21	$\quad\quad A$	12 14-15 16-20 $\vee$ E
22	$\quad \Box A$	12-21 $\Box$ I
23	$\Box A \vee \Diamond B$	2-22 $\sim$ E
24	$\Box(A \vee B) \supset (\Box A \vee \Diamond B)$	1-23 $\supset$ I

Exercise. Prove the rest of the results. (Use of well-known derived rules of *SD* will simplify the derivations.)

4.2 Strict Implication

We saw in a previous module that necessity-sentences can be understood semantically as mirroring in the object-language the meta-logical property of validity. The same kind of mirroring relation holds between a "strict implication" α ⥽ β and the meta-logical relation of semantic entailment. The original modal axiom systems of C. I. Lewis were aimed at producing theorems that were supposed to capture the nature of "implication." In a later book, Lewis contrasted the way modal logic treats deduction with the way it is treated in the "truth-value" system *SI*.

> The principles of Strict Implication express the facts about any such deduction in an explicit manner in which they cannot be expressed within the truth-value system itself, for the reason that, in Strict Implication, what is tautological is distinguished from what is merely true, whereas this difference does not ordinarily appear in the symbols of the truth-value system. (*Symbolic Logic*, 1932, p. 247)

The "tautological" is expressed by the necessity-sentence in modal logic but cannot be expressed in non-modal Sentential Logic. It can only be expressed in the meta-language of *SL*.

Lewis wanted to treat formally the relation of semantic entailment. This much is clear from his early discussion of the meaning of strict implication.

> The strict implication, $p \prec q$, means, "It is impossible that p be true and q be false," or "p is inconsistent with the denial of q."... Any set of mutually consistent propositions may be said to define a "possible situation" or "case" or "state of affairs." And a proposition may be "true" of more than one such possible situation—may belong to more than one such set.... In these terms, we can translate $p \prec q$ by "Any situation in which p should be true and q false is impossible." (*A Survey of Symbolic Logic*, 1918,, pp. 332-333.)

From the standpoint of our formal semantics, we are interested in the case of semantic entailment where the entailing set of sentences contains only one member:

$\{\gamma\} \vDash_{KI} \alpha$.

For example, we know that

$\{P \wedge Q\} \vDash_{KI} P$.

This entailment holds because at any world **w** on any interpretation **I**, if '$P \wedge Q$' is true at **w** then 'Q' is true there as well.

We can say generally that if $\{\alpha\} \vDash_{KI} \beta$, then $\vDash_{KI} \alpha \prec \beta$.

Proof. If $\{\alpha\} \vDash_{KI} \beta$, then for all worlds **w** in all interpretations **I** based on any *KI*-frame, if $\mathbf{v_I}(\alpha, \mathbf{w}) = \mathbf{T}$, then $\mathbf{v_I}(\beta, \mathbf{w}) = \mathbf{T}$. Therefore, if a world $\mathbf{w}_i$ is accessible to a given world **w**, if $\mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T}$, then $\mathbf{v_I}(\beta, \mathbf{w}_i) = \mathbf{T}$. So for all worlds **w** and all interpretations **I** in all *KI*-frames, $\mathbf{v_I}(\alpha \prec \beta, \mathbf{w}) = \mathbf{T}$, which is to say that $\vDash_{KI} \alpha \prec \beta$.

Inspection of the derivational rule SR-$\prec$ shows that if $\{\alpha\} \vdash_{KD} \beta$, then $\vdash_{KD} \alpha \prec \beta$. One need only make an assumption of α to the right of a restricted scope line, derive β there and use SR-$\prec$ to discharge the assumption and write $\alpha \prec \beta$.

These results establish that strict implication in *K* represents in the object-language the meta-logical relation of semantical entailment when it holds. It is important to note that what counts as a semantical entailment is different relative to different semantical systems. Exactly which set of valid strict implications is produced by a semantical system is vital to its adequacy as a logic of "implies."

On the other hand, the converse of this result does not hold for the *K*-systems. That is, it is not the case that if $\vDash_{KI} \alpha \prec \beta$, then $\{\alpha\} \vDash_{KI} \beta$. Consider a frame **Fr** which contains a world **w** which is not accessible to itself. Let $\alpha \prec \beta$ and α have the value **T** at **w**. We may assign the value **F** to β at **w**, since the truth of $\alpha \prec \beta$ has no consequences for **w**, but only for worlds accessible to **w**.

This result is flatly incompatible with Lewis's view that when $\alpha \prec \beta$, any situation in which α is true and β false is impossible. So we should say that the *K*-systems are too weak to express everything about strict implication that Lewis wanted it to express. The systems *S2* and *S3*, with which Lewis was working, are such that for every valid strict implication, the corresponding consequence relation holds.

Strict implication can be understood as a necessitated material conditional. $\alpha \strictif \beta$ is semantically equivalent to $\Box(\alpha \supset \beta)$. **SR**-$\strictif$ states that if either $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**F** or $\mathbf{v_I}(\beta, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$, then $\mathbf{v_I}(\alpha \strictif \beta, \mathbf{w})$=**T**; $\mathbf{v_I}(\alpha \strictif \beta, \mathbf{w})$=**F** otherwise. By **SR**-$\supset$, if either $\mathbf{v_I}(\alpha, \mathbf{w})$=**F** or $\mathbf{v_I}(\beta, \mathbf{w})$=**T**, then $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w})$=**T**; $\mathbf{v_I}(\alpha \supset \beta, \mathbf{w})$=**F** otherwise. Finally, **SR**-$\Box$ states that if $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T** at all worlds $\mathbf{w}_i$ in **I** such that $\mathbf{Rww}_i$, then $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**T**; $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**F** otherwise. The condition which must be satisfied at $\mathbf{w}_i$ for the truth of both $\alpha \strictif \beta$ and $\Box\alpha$ is exactly the same.

The same result holds in the derivational system. If $\alpha \supset \beta$ is derivable from no assumptions in a restricted scope line, we can end the scope line and affix a '$\Box$' to the front of it. And this can be done just in case we can derive β from the Modal Assumption α and no other assumptions, which entitles us to end the restricted scope line and write $\alpha \strictif \beta$.

Because strict implication can be treated as a necessitated material conditional, there is a result of closure that is of interest. Since $\{\alpha, \alpha \supset \beta\} \vdash_{KD} \beta$, it follows by Closure that $\{\Box\alpha, \Box(\alpha \supset \beta)\} \vdash_{KD} \Box\beta$, in which case $\{\Box\alpha, \alpha \strictif \beta\} \vdash_{KD} \Box\beta$. (And the corresponding result holds for the axiomatic and semantical systems.) This result may be proved directly by way of a derivation.

To prove: $\{\Box\alpha, \alpha \strictif \beta\} \vdash_{KD} \Box\beta$

1		$\Box\alpha$	Assumption
2		$\alpha \strictif \beta$	Assumption
3	$\Box$	α	1 SR-$\Box$
4		$\alpha \supset \beta$	3 SR-$\strictif$
5		β	3 4 $\supset$ E
6		$\Box\beta$	3-5 $\Box$ I

Exercise. Give a derivation-schema that shows the analogue of this result for the '$\Diamond$' operator.

The result just given shows once again how strict implication represents the consequence relation in *KI* and *KD*. In *KD*, for example, we have by the closure property that if $\{\alpha\} \vdash_{KD} \beta$, then $\{\Box\alpha\} \vdash_{KD} \Box\beta$. The antecedent of this conditional can be thought of as being represented by the strict implication sentence-schema in $\{\Box\alpha, \alpha \strictif \beta\} \vdash_{KD} \Box\beta$. Indeed, we might think of this schema as representing in a sense the "closure" of necessity under there relation of strict implication. The property of being necessary is not lost when inference is made by way of strict implication.

4.3 Other Modal Operators

In Module 3, a number of other modal operators were discussed in an informal way. Now that we have our formal modal systems in hand, we may treat them more rigorously.

Strict implication can be understood as the result of prefixing a modal operator to (or "modalizing") Sentential Logic sentences of the form $\alpha \supset \beta$ to get $\Box(\alpha \supset \beta)$. Other *SL* sentence-forms are subject to the same procedure. A modalized *SL* form can serve as the basis for defining other modal operators that have appeared in the literature of modal logic. Here is a list of all the possibilities.

$\Box{\sim}\alpha$	$\Diamond{\sim}\alpha$
$\Box(\alpha \wedge \beta)$	$\Diamond(\alpha \wedge \beta)$
$\Box(\alpha \vee \beta)$	$\Diamond(\alpha \vee \beta)$
$\Box(\alpha \supset \beta)$	$\Diamond(\alpha \supset \beta)$
$\Box(\alpha \equiv \beta)$	$\Diamond(\alpha \equiv \beta)$

Three of these are of historical interest and will be examined here: $\Box(\alpha \vee \beta)$ (which is equivalent to $\alpha \veebar \beta$), $\Diamond(\alpha \wedge \beta)$ (which is equivalent to $\alpha \circ \beta$), and $\Box(\alpha \equiv \beta)$ (which is equivalent to $\alpha \leftrightarrows \beta$).[6]

4.3.1 Intensional Disjunction

A modal operator that can be formed by modalizing an *SL* sentence-form is *intensional disjunction*: $\Box(\alpha \vee \beta)$. Lewis started his investigations into modal logic with a discussion of this operator (though without symbolizing it).[7] In *A Survey of Symbolic Logic*, he used a '∨' for intensional disjunction and a '+' for truth-functional disjunction. Here is how he described the difference.

> Both $p \vee q$ and $p+q$ would be read as "Either p or q". But $p \vee q$ denotes a *necessary* connection; $p+q$ a merely factual one. Let p represent "Today is Monday", and q, "2 + 2 = 4". Then $p+q$ is true but $p \vee q$ is false. In point of *fact*, at least one of the two propositions, "Today is Monday", and "2 + 2 = 4", is true; but there is no *necessary* connection between them. "Either . . . or . . ." is ambiguous in this respect. Ask the members of any company whether the proposition "Either today is Monday or 2 + 2 = 4" is true, and they will disagree. Some will combine "Either . . . or . . ." to the $p \vee q$ meaning, others will make it include the $p+q$ meaning; few, or none, will make the necessary distinction. (*A Survey of Symbolic Logic*, p. 294)

An example of the kind of sentence Lewis had in mind as a necessary disjunction is (in the notation of this text) $\alpha \vee \sim\alpha$. Sentences with this form are valid in *KI*, and by necessitation, $\Box(\alpha \vee \sim\alpha)$ is valid in *KI* as well.

4.3.2 Consistency

Lewis showed a number of properties of consistency (in his system *S3*), including these: "If p and q are consistent, then q and p are consistent", "If p and q are consistent, then it is possible that p be true", "If it is possible that p be true, then p is consistent with itself" (*A Survey of Symbolic Logic*, p. 300). Viewed as consequence-relations or strict implications, these hold in *KI* and *KD* as well.

Exercise. Symbolize these conditionals as strict implications. Show that they hold in *KI*.

Of some interest is a consequence relation of *S3* (and *KD*) discussed by Lewis. Here we give it in its semantical form in *KI*:

$$(\alpha \circ (\beta \wedge \gamma)) \vDash_{KI} (\beta \circ (\alpha \wedge \gamma)).$$

We can see that this holds because it is equivalent to

$$\Diamond(\alpha \wedge (\beta \wedge \gamma)) \vDash_{KI} \Diamond(\beta \wedge (\alpha \wedge \gamma)).$$

The non-modal conjunctions inside the possibility-operators are clearly equivalent, as can be seen from the Sentential Logic derivational rule of Association. But the consistency relation itself is not associative in *K*.

$$\{\alpha \circ (\beta \circ \gamma)\} \nvDash_{KI} \beta \circ (\alpha \circ \gamma).$$[8]

This was recognized by Lewis, who remarked that because of the failure, the treatment of consistency "seems incomplete" (*Survey*, p. 300). The reason for the failure can be seen quite readily when an equivalent form is displayed.

$$\{\Diamond(\alpha \wedge \Diamond(\beta \wedge \gamma))\} \nvDash_{KI} \Diamond(\Diamond(\beta \wedge \alpha) \wedge \gamma).$$

The problem is that two distinct possibilities do not imply a conjunctive possibility. In the present case,

[6] Prefixing a '□' to a negated sentence yields impossibility, which will not be given a formal treatment here.

[7] "Implication and the Algebra of Logic", *Mind* (New Series) 84, October, 1912.

[8] A slash through a turnstile or double-turnstile indicates that the property or relation indicated by the symbol does not hold.

$\Diamond(\beta \wedge \gamma) \nvDash_{KI} \Diamond\beta \wedge \Diamond\gamma)$ If that entailment did hold, then both sentences would be reduced to conjunctions, and the rule of Association would apply to them.

uppose that we have an interpretation **I** with worlds **w**, $\mathbf{w}_1$, and $\mathbf{w}_2$, and that $\mathbf{Rww}_1$ and $\mathbf{Rw}_1\mathbf{w}_2$. But suppose further that it is not the case that $\mathbf{Rww}_2$ or that any other accessibility relation holds. Let α be true and γ false at $\mathbf{w}_1$. At $\mathbf{w}_2$, let β and γ both be true. In that case, $\beta \wedge \gamma$ is true at $\mathbf{w}_2$, and so $\Diamond(\beta \wedge \gamma)$ is true at $\mathbf{w}_1$, in which case, so is $\alpha \wedge \Diamond(\beta \wedge \gamma)$. Then $\Diamond(\alpha \wedge \Diamond(\beta \wedge \gamma))$ is true at **w**. Now let γ be false at $\mathbf{w}_1$, in which case $\Diamond(\beta \wedge \alpha) \wedge \gamma$ is false at $\mathbf{w}_1$. Since $\mathbf{w}_1$ is the only world accessible to **w**, $\Diamond(\Diamond(\beta \wedge \alpha) \wedge \gamma)$ is false there as well, which was to be demonstrated. Unlike the other results, this failure is not overcome in the stronger systems we will be considering. No restrictions on accessibility will block the kind of counter-example just given. Non-associativity seems to be a fundamental fact about consistency.

4.3.3 Strict Equivalence

Lewis explained strict equivalence $\alpha \mathbin{\varepsilon\!\!\ni} \beta$ as a relation that "denotes an equivalence of logical import or meaning, while $p{\equiv}q$ denotes simply an equivalence of truth-value" (*Survey of Symbolic Logic*, p. 294). He could have added that a strict equivalence between α and β can be understood as the result of prefixing a necessity operator to a biconditional: $\Box(\alpha \equiv \beta)$. In our semantics, a sentence of this form is true at a world just in case the embedded biconditional is true at all accessible worlds. That is, at each accessible world, either α and β are both true there or they are both false there. As can be seen, this is exactly what makes a strict equivalence true.

Exercise. Give a derviation to prove the following: $\{(\alpha \dashv\!\!\!\exists\, \beta) \wedge (\beta \dashv\!\!\!\exists\, \alpha)\} \vdash_{KD} \Box(\alpha \equiv \beta)$.

The same result can be obtained by appeal to closure. The conjunction $(\alpha \dashv\!\!\!\exists\, \beta) \wedge (\beta \dashv\!\!\!\exists\, \alpha)$ is equivalent to $\Box(\alpha \supset \beta) \wedge \Box(\beta \supset \alpha)$. We know from Sentential Logic that $\alpha \supset \beta$ and $\beta \supset \alpha$ are both consequences of $(\alpha \equiv \beta)$. By closure, $\Box(\alpha \supset \beta)$ and $\Box(\beta \supset \alpha)$ are both consequences of $\Box(\alpha \equiv \beta)$. And $\Box(\alpha \supset \beta) \wedge \Box(\beta \supset \alpha)$ is a consequence of these two sentence forms. So $\Box(\alpha \supset \beta) \wedge \Box(\beta \supset \alpha)$ is a consequence of $\Box(\alpha \equiv \beta)$.

Conversely, by closure, $\Box(\alpha \equiv \beta)$ is a consequence of $\Box(\alpha \supset \beta)$ and $\Box(\beta \supset \alpha)$. And these two sentences are consequences of $\Box(\alpha \supset \beta) \wedge \Box(\beta \supset \alpha)$. Therefore, $\Box(\alpha \equiv \beta)$ is a consequence of $\Box(\alpha \supset \beta) \wedge \Box(\beta \supset \alpha)$ It follows that $\Box(\alpha \supset \beta) \wedge \Box(\beta \supset \alpha)$ and $\Box(\alpha \equiv \beta)$ are consequences of each other. That is, they are equivalent forms.

Note also that single possibility- and necessity-sentences are, in the sense discussed in the section on strict implication, "closed" under strict equivalence. In the semantical system, this means that $\{\Box\alpha, \alpha \mathbin{\varepsilon\!\!\ni} \beta\} \vdash_{KD} \Box\beta$ and $\{\Diamond\alpha, \alpha \mathbin{\varepsilon\!\!\ni} \beta\} \vdash_{KD} \Diamond\beta$. This follows from the fact that strict equivalence-sentences are conjunctions of strict implication sentences, for which closure holds.

5 Non-Consequences in *K* and its Equivalents

The semantical system *KI* and its derivational twin *KD* are very weak. A number of consequence relations which one might expect to hold, given an intended informal interpretation of the logic, do not. These failures mean that the system is inadequate for those intended uses of modal logic. Before turning to these applications, we will look at some failures of consequence that would seem to present a problem for most applications. We will express these failures primarily in terms of the semantical system *KI*.

Because a *KI*-frame may have any relation of accessibility holding of its worlds, some *KI*-frames will contain "dead ends," worlds to which no world is accessible. On any interpretation based on such a frame **Fr**, a sentence of the form $\Diamond\alpha$ will be false at such a dead-end. Therefore,

For all α and some frame **Fr**, $\nvDash_{\mathbf{Fr}} \Diamond\alpha$.

And since such a frame **Fr** is a member of the class of all *KI*-frames, we may assert:

For all $\alpha, \nvDash_{KI} \Diamond\alpha$.

Because of soundness, the corresponding failure occurs in the derivational system. An examination of the rules shows that there is no rule for the introduction of a possibility operator. Adding such a rule will allow theorems of the form $\Diamond\alpha$. The failure is overcome in system *D*, to be discussed in a later module. A related failure is that a necessity-sentence $\Box\alpha$ does not have as a consequence the possibility-sentence $\Diamond\alpha$. This failure was shown in the previous module.

Another significant non-consequence is this:

$\Box\alpha \nvDash_{KI} \alpha$.

A necessity-sentence does not have its embedded sentence as a consequence. The reason here is that in some frames, there are worlds that are not accessible to themselves. This allows an interpretation **I** on whose valuation function the embedded sentence α fails to hold at the world where it is evaluated but does hold at all accessible worlds. The accesibility relation need not be reflexive on a *KI*-frame. Thus we can construct a frame with a world **w** such that it is not the case that **Rww** and on which $\mathbf{v_I}(\alpha,\mathbf{w})$=**F**, while for all accessible $\mathbf{w}_i$, $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=**T**. On that intrepretation, $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=**T**, which blocks the semantical entailment. We shall see that system *T* will allow this consequence. The derivational system for *T* will include a rule of $\Box$ Elimination which will directly yield the consequence in question.

Detachment for strict implication also fails in *KI*.

$\{\alpha, \alpha \dashv\!\!\!3\, \beta\} \nvDash_{KI} \beta$.

This is most readily seen in the derivational system. If a sentence α occurs on a line of a derivation, and $\alpha \dashv\!\!\!3\, \beta$ occurs on another line, one may not infer β. One would have to initiate a strict scope line and use SR-$\dashv\!\!\!3$ to get $\alpha \supset \beta$ there, but α may not have a form which would allow the use of Strict Reiteration across that line. From a semantical standpoint, the reason is that there may be an assignment of **T** to α and **F** to β at a world **w**, despite the fact that $\alpha \dashv\!\!\!3\, \beta$ is assigned **T** there. This can occur if **w** is not accessible to itself. While at all accessible worlds where α is true, β must be assigned **T**, **w** is not one of those worlds, and so it is permissible that β be assigned **F** there. System *T* will yield detachment for strict implication.

It might be thought that what is necessary is necessarily necessary, but

$\{\Box\alpha\} \nvDash_{KI} \Box\Box\alpha$.

This failure is due to the fact that the accessibility relation on a *KI*-frame does not have to be transitive. Based on frames where $\mathbf{Rww}_1$ and $\mathbf{Rw}_i\mathbf{w}_j$, but not $\mathbf{Rww}_j$, interpretations **I** can be given on which $\mathbf{v_I}(\Box\alpha,\mathbf{w})$=**T** and $\mathbf{v_I}(\Box\Box\alpha,\mathbf{w})$=**F**. Consider a frame on which $\mathbf{Rww}_i$ and $\mathbf{Rw}_i\mathbf{w}_j$, but nothing else, and an interpretation **I** based that frame. Let $\mathbf{v_I}(\alpha,\mathbf{w}_i)$=**T** and $\mathbf{v_I}(\alpha,\mathbf{w}_j)$=**F**. Since $\mathbf{w}_i$ is the only world accessible to **w**, $\mathbf{v_I}(\Box\alpha,\mathbf{w})$=**T**. But $\mathbf{v_I}(\Box\alpha,\mathbf{w}_i)$=**F**, since $\mathbf{v_I}(\alpha,\mathbf{w}_j)$=**F** at the only accessible world. So $\mathbf{v_I}(\Box\Box\alpha,\mathbf{w})$=**F**, which was to be proved. In the next chapter, we shall see how system *S4* allows this consequence. The derivational system for *S4* will include a more liberal rule of Strict Reiteration.

An even stronger thesis is that what is possible is necessarily possible:

$\{\Diamond\alpha\} \nvDash_{KI} \Box\Diamond\alpha$.

Here the result follows from the fact that a *KI*-frame need not be euclidean. A relation is euclidean when, if $\mathbf{Rww}_i$ and $\mathbf{Rww}_j$, then $\mathbf{Rw}_i\mathbf{w}_j$. So consider a frame in which $\mathbf{Rww}_i$ and $\mathbf{Rww}_j$, and nothing else. Let $\mathbf{v_I}(\alpha,\mathbf{w}_i)$=**T**. Then $\mathbf{v_I}(\Diamond\alpha,\mathbf{w})$=**T**. Now let $\mathbf{v_I}(\alpha,\mathbf{w}_j)$=**F**. Then $\mathbf{v_I}(\Diamond\alpha,\mathbf{w}_i)$=**F**. So $\mathbf{v_I}(\Box\Diamond\alpha,\mathbf{w})$=**F**, which was to be proved. System *B* overcomes this counter-example. As with *S4*, a more liberal Strict Reiteration rule will be added to get a derivational system for *B* to allow the consequence.

In the general case, closure of possibility over the consequence relation does not hold in *K* or any stronger system based on it. That is,

It is not the case that if $\{\gamma_1, \ldots, \gamma_n\} \vDash_{KI} \alpha$, then $\{\Diamond\gamma_1, \ldots, \Diamond\gamma_n\} \vDash_{KI} \Diamond\alpha$.

Proof. It is enough to show that the consequence does not hold in a single instance, $\{\Diamond A, \Diamond B\} \nvDash_K \Diamond(A \wedge B)$, where $\{A, B\} \vDash_{KI} A \wedge B$. Consider a *KI*-frame Fr=<W,R> where W={w, w_1, w_2} and Rww_1, Rww_2. Let I be based on Fr, with the following assignments: $\mathbf{v}_I(A, w_1)$=**T**, $\mathbf{v}_I(B, w_1)$=**F**, $\mathbf{v}_I(A, w_2)$=**F**, and $\mathbf{v}_I(B, w_1)$=**T**. Now $\mathbf{v_I}(\Diamond A, w)$=**T** and $\mathbf{v}_I(\Diamond B, w)$=**T**, by **SR**-$\Diamond$. However, it is also the case that $\mathbf{v}_I(A \wedge B, w_1)$=**F** and $\mathbf{v}_I(A \wedge B), \mathbf{w}_2)$=**F**. Therefore, $\mathbf{v}_I(\Diamond(A \wedge B), w)$=**F**. No standard restriction on accessibility will prevent a counter-example to be formulated, since nothing forces the assignment of specific truth-values to sentence-letters at a given world.