Module 6
Applications of Modal Propositional Logic

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Now that the K-systems have been investigated, we are in a position to discuss their adequacy for representing modal sentences of natural language that we think to be true, and for representing relations of logical consequence among modal sentences. The philosophical interpretations of the modal operators will be called *applications* of the respective formal systems. In this module, we shall examine the adequacy of the K-systems as a logic of truth (alethic modal logic), of implication (one kind of conditional logic), of obligation and permission (deontic logic), of belief (doxastic logic), of knowledge (epistemic logic), and of time (temporal logic). The general result will be that the K-systems are a reasonable starting-point for all such applications. But in each case, stronger systems are needed to represent all the truths and consequence relations we want to assert using them.

1 Alethic Modal Logic

The most common way of understanding our modal operators is as expressing ways or modes in which a sentence can be true. A truth may be necessary, possible, or contingent, depending on its structure and how it is interpreted. Modal logic understood as being about the different modes of truth is called "alethic," after the Greek word for truth, '*aletheia*.'

1.1 A Logic of Truth-Values

It is natural to think of the K-systems as being formal logics of truth. In the formal semantics $S(K)$, we interpret sentences of $\mathscr{L}_0$ as having one of two values, t or f, but not both. The underlying non-modal system, $S(0)$, is intended to capture generally-shared beliefs about the nature of truth and falsehood. The property of Bivalence, that every sentence $\mathscr{L}_0$ has one of only two values, is widely (though not universally) held to apply to truth and falsehood of sentences of natural language. The property of Truth-Functionality, that no sentence of $\mathscr{L}_0$ have more than one value, reflects a view held even more widely (though again, not universally), that every sentence of natural language is true is not false, and *vice-versa*.

Moreover, the semantical rules for non-modal sentences conform to other of our beliefs about the properties of what is true and what is false, e.g., that a sentence is true if and only if its negation is false. As has been noted, however, that the convenient semantical rule for the '$\supset$' does not capture any kind of necessary connection between the truth of the antecedent and the truth of the consequent. So we can understand $S(0)$ as yielding a logic of truth, but it may be one that does not say as much about truth as we would like. Lewis held this view, and it spurred him to develop modern modal logic.

What Lewis showed is that modal logic allows the expression in a formal way of some of the properties of the underlying logic of truth-values $S(0)$. On this reading, we take t and f to indicate truth and falsehood, and we understand the '$\Box$' to indicate necessary truth and the '$\Diamond$' to indicate possible truth. In this way, we provide an enriched logic of truth that can make distinctions, within its formal language $\mathscr{L}$, that cannot be made in $\mathscr{L}_0$. For example, it allows a representation of "strict implication" that may better reflect the way we understand some conditional sentences of English. It also allows the representation of a kind of disjunction, "intensional" disjunction, that may satisfy those who think that "Either today is Monday or $2 + 2 = 4$" is not a true sentence.

1.2 What Are Necessary and Possible Truth?

But once we have arrived at this point, things become somewhat murky. In $\mathscr{L}$ we have a language more expressive that $\mathscr{L}_0$, which has been given a formal semantics that might be used to give us a logic of necessary and possible truth. We want to know, of course, whether the semantics is adequate for this purpose. But then we face the questions: What is necessary truth? What is possible truth?

Historically, different answers have been given to these question, and it turns out that no single system of modal logic is adequate to capture all the different ways of understanding necessary and possible truth. This consideration motivates the study of a number of different modal systems as potential logics of necessary and possible truth. Here is an excellent description of the situation in Hughes and Cresswell's 1968 *An Introduction to Modal Logic*.

> This multiplicity of systems is apt to provoke the question, Which system is the correct one? Now the assumption behind this question seems to be that we have in mind some single sense of 'necessity' and 'possibility,' and that systems weaker than the correct one will give us less than the whole truth, while stronger systems will contain theses which even if plausible are really false. But perhaps the systems are not rivals in this way. It is at least possible that a number of systems may each give us the truth about necessity and possibility, though each in a somewhat different sense of those terms. Merely constructing semantic models will not by itself give us an adequate characterization of these different senses; for that a great deal of intricate philosophical work would be required, though, as we have seen the semantic models can give us valuable help in this task. (p. 79)

Our present interest is to get some idea of whether the systems presented here are good formalizations of necessity and possibility in different senses of those terms. In what follows, we will do a very small portion of the intricate philosophical work required.

1.3 Varieties of Necessity and Possibility

It is common to rank the alethic modalities in a hierarchy. At the most abstract level, there are the "logical" alethic modalities: logical necessity and logical possibility. Below this lie "metaphysical" possibility and necessity. At the most concrete level of the hierarchy are the "physical" alethic modalities.

Standardly, all logical necessities are taken to be metaphysically necessary, and all metaphysical necessities are said to be physically necessary. The same relation holds between metaphysical and physical necessities. And by transitivity, we have it that whatever is logically necessary is physically necessary as well. Given that some metaphysical necessities are not logical necessities, the class of logically necessary truths is thought to be smaller than that the class of metaphysically necessary truths. And for the same kind of reason, the class of metaphysically necessary truths is said to be smaller than the class of physically necessary truths.

We can represent this relation schematically by creating a new *metalogical* symbol which adds one of 'L,' 'M,' and 'P' as a subscript to the '$\Box$' operator, treated as a name of itself. '$\Box_L\varphi$' thus states metalogically that φ is logically necessary, etc. Given the claims made in the last paragraph, we have the following relations:

If $\Box_L\varphi$, then $\Box_M\varphi$,

If $\Box_M\varphi$, then $\Box_P\varphi$, If $\Box_{\mathrm{L}}\varphi$, then $\Box_{\mathrm{P}}\varphi$,

It is not the case that if $\Box_{\mathrm{M}}\varphi$, then $\Box_{\mathrm{L}}\varphi$,

It is not the case that if $\Box_{\mathrm{P}}\varphi$, then $\Box_{\mathrm{M}}\varphi$.

1.4 Logical Necessity and Possibility

Before we can determine how well K and its equivalents represent logical necessity and possibility, we need to have some informal characterization of what these modalities are.

1.4.1 Logical Necessity

A good starting-point might be that φ is logically necessary if and only if it is true by virtue of its logical form, or by virtue of the "laws of logic."[1] We will use these characterizations interchangeably. These characterizations of the logical modalities can be made rigorous by tying them to the rules of a semantical, derivational, or axiomatic system. We might want to say, then, that φ is logically necessary just in case it is valid in some semantical system, or that it is a theorem of a derivational or axiomatic system.

Now we are in a position to ask whether the K-systems are the ones which embody just the right laws of logic. (From now on, we will use the semantical system $S(K)$ as a proxy for the other systems.) First, we will investigate whether validity in $S(K)$ is a sufficient condition for logical necessity:

If $\vDash_{S(K)} \varphi$, then $\Box_{\mathrm{L}}\varphi$.

This schematic representation can be fleshed out on the basis of Necessitation:

If $\vDash_{S(K)} \varphi$, then $\vDash_{S(K)} \Box\varphi$.

Reverting to our informal characterization of logical necessity, we can say that if φ is true by virtue of the "laws" of the K-systems, then $\Box\varphi$ is also true by virtue of the "laws" of the K-systems. Note that this does not imply that $\Box\varphi$ is true in some absolute sense. The latter statement can be made only outside the system, along the lines of:

If $\vDash_{S(K)} \Box\varphi$, then $\Box\varphi$ is true.

The heart of Quine's objection to modal logic is his view that the statement of the necessary truth of a sentence can only be made in the meta-language. Thus, however suggestive sentences of the form $\Box\varphi$ might be of a statement about the necessity of a sentence, it is not such a statement. We shall proceed conservatively and say only that what '$\vDash_{S(K)} \Box\varphi$' and its cousins indicate is φ has a logical form that makes it the schema of a necessarily true sentence.

If we agree that the valid sentences and theorems of $S(0)$ and $S(K)$, respectively, are logical truths, then at least one minimal condition is met in the K-systems, since:

If $\vDash_{S(0)} \varphi$, then $\vDash_{S(K)} \varphi$.

Aside from their being extensions of the $\mathscr{L}_0$-systems, the K-systems share the property of closure of the '$\Box$' operator over logical implication, as in $S(0)$.

[1] If we characterize logically necessary truth as what cannot possibly be false, we would either have to define possibility independently of necessity or give a reciprocal characterization of the two. Perhaps possibility could be understood independently as self-consistency. But this is a matter of logical form, and so we are back to the proposed characterization.

If $\{\gamma_1, \ldots, \gamma_n\} \vDash_{S(K)} \varphi$, then $\{\Box\gamma_1, \ldots, \Box\gamma_n\} \vDash_{S(K)} \Box\varphi$.

(Note that what counts as "logical implication" is relative to the system. In the present case, it is $S(K)$-entailment.) So if, by virtue of the "laws of logic," the truth of each of the γ_is guarantees the truth of φ, then the modalized γ_is, $\Box\gamma_1, \ldots, \Box\gamma_n$, guarantee the truth of $\Box\varphi$. If we take the $\Box\gamma_i$ to mean that γ_i is true by virtue of its logical form, then given the "logical implication" of φ, we ought to take it that φ is also true by virtue of its logical form. What follows from what is true by its logical form is itself true by its logical form. So we shall accept closure of this kind as characteristic of logical necessity.

Now let us turn to the question of whether validity in $S(K)$ is a necessary condition for logical necessity. Is it the case that:

If $\Box_{\mathrm{L}}\varphi$, then $\vDash_{S(K)} \varphi$?

There are several reasons to think not. One seems very urgent. We want to say that if φ is necessarily true by virtue of the "laws of logic," then φ is true. However, as we saw in the last module:

$\{\Box\varphi\} \nvDash_{S(K)} \varphi$.

This seems to indicate that given that φ is true by its logical form, it does not follow in the K-systems that φ is true. There are some models on some frames in which $\Box\varphi$ is true at a world but φ is not true there. The closest we get to the desired result is:

If $\vDash_{S(K)} \Box\varphi$, then $\vDash_{S(K)} \varphi$.

If the system makes it impossible for $\Box\varphi$ to be false, then it makes it impossible for φ to be false as well, in which case φ is true.

A second apparent reason to think that $S(K)$-validity is not sufficient for logical necessity is that $S(K)$ does not validate the iteration of modalities.

$\{\Box\varphi\} \nvDash_{S(K)} \Box\Box\varphi$.

It might seem that if φ is true in virtue of its logical form, then it is true in virtue of its logical form that φ is true in virtue of its logical form. (Though one might instead hold that the logical form that makes φ true need not be the same form that makes $\Box\varphi$ true.) Be that as it may, the K-systems have a feature that may satisfy the demand for some kind of iterative principle. By necessitation we have:

If $\vDash_{S(K)} \varphi$, then $\vDash_{S(K)} \Box\varphi$, and

If $\vDash_{S(K)} \Box\varphi$, then $\vDash_{S(K)} \Box\Box\varphi$, so

If $\vDash_{S(K)} \varphi$, then $\vDash_{S(K)} \Box\Box\varphi$.

This may well be all we wish to say about iteration. If φ is true by the laws of the system, then so is $\Box\varphi$, as is $\Box\Box\varphi$, etc.

1.4.2 Logical Possibility

Given that a necessarily true sentence is a sentence which is true by virtue of its logical form, how should we understand a possibly true sentence? We might wish to say that its logical form permits it to be true, or that it is consistent with its logical form that it be true. Yet another way to understand logical possibility is that it is not the case that its logical form requires that it be false. All these ways of understanding possible truth might be expressed through the notion of semantical consistency. We might wish to say that:

If $\{\varphi\}$ is satisfiable in $S(K)$, then $\Diamond_{\mathrm{L}}\varphi$.

A sentence φ is satisfiable in $S(K)$ if and only if there is no world in any $S(K)$-frame at which φ is false on some model. This seems to be at least a necessary condition for the possible truth of φ.

Again, we might ask whether the condition is sufficent, and here the same problems arise as with the condition for necessity discussed above. In the first place, $\{\sim\Diamond\varphi, \varphi\}$ is satisfiable in $S(K)$. There is a frame which contains a world at which, for some model, φ is true and $\Diamond\varphi$ is false (since $\sim\Diamond\varphi$ is true). It seems, however, that what is true is at least possibly true. Secondly, $\{\Diamond\Diamond\varphi, \sim\Diamond\varphi\}$ is satisfiable in $S(K)$. We might want to say that if $\Diamond\varphi$ has a logical form that permits it to be true, φ has a form that permits it to be true.

If we have both modal operators at our disposal, we might think that the following results are reasons why the semantical system $S(K)$ is not sufficient to represent logical necessity and possibility. The first result is that what is necessarily true need not be possibly true.

$\{\Box\varphi\} \nvDash_{S(K)} \Diamond\varphi$.

It is difficult to see why a sentence which must be true by virtue of its mere form is not such that it is permitted to be true by that form.

A less clear-cut case is the following non-entailment:

$\{\Diamond\Box\varphi\} \nvDash_{S(K)} \Box\varphi$.

We might wish to say that if the logical form of a necessity-sentence $\Box\varphi$ allows the sentence to be true, then $\Box\varphi$ itself is true by virtue of its logical form.

1.5 Assessment of the Adequacy of the *K*-Systems for Representing Logical Modalities

It appears that the *K*-systems are too weak to represent some of the properties of logical necessity and possibility. Specifically, we want to say that what is logically necessary is true, and that what is true is logically possible. To get these results, we need stronger systems.

A general objection to the use of $S(K)$ to represent the logical modalities is that it relativizes the determination of the truth-value of a modal sentence to accessible worlds rather than determining them by reference to the set of all worlds (in a frame). Leibniz is credited with the leading idea of possible-worlds semantics, that a sentence is necessarily true if and only if it is true at all possible worlds. One version of the semantics for *S5* is based directly on this idea, and it might be thought better suited to the idea of logical possibility and necessity. Then we can say that a sentence of the form $\Diamond\varphi$ is true, period, just in case φ is true at some possible world, period. And we can say that a sentence of the form $\Box\varphi$ is true, period, just in case φ is true at all possible worlds, period. As will be seen in a later module, the semantics for *S5* can be given in this way.

Kenneth Konyndyk, for example, claims that "broadly logical necessity" is best captured by *S5*.

> That which is broadly logically necessary is true in all possible worlds, true no matter what. When we use this picture of possible worlds and accessibility relationships, the necessity of a proposition in a given world—in the actual world, for example—is the truth of that proposition in all possible worlds accessible to the actual world. But this would not coincide with truth in all possible worlds unless the accessibility relation were reflexive, symmmetric, and transitive. So if we think of broadly logically necessary

truths as true in *all* possible worlds, S5 seems to be the most adequate modal system. (*Introductory Modal Logic*, p. 60)

1.6 Hypothetical Necessity and Possibility

We might want to distinguish between sentences that are true by virtue of their logical form and those which are not. If a sentence is not true due to its logical form (or false by virtue of its logical form), it is called *contingent*. The word 'contingent' in its most basic sense indicates a dependence on something else. In the present case, the truth of contingent φ at a world depends on the way that world is.

At first blush, it seems that being a necessary truth does not depend on any conditions. But it is quite coherent to say that a sentence is necessarily true given that some conditions are met. We then say that the sentence is "hypothetically necessary."[2] This is one way of treating metaphysical and physical necessity. A sentence φ is metaphysically necessary, it might be held, just in case its truth is guaranteed by the truth of some other sentences. In the case of metaphysical necessity, they might be metaphysical "first principles." In that of physical necessity, these sentences might be "laws of nature."

Corresponding to hypothetical necessity is hypothetical possibility, what is possible under a given condition. Thus, what is metaphysically possible is what is possible given some set of first principles, and what it physically possible is what is possible given a set of laws of nature.

The semantical system $S(K)$ seems well-suited as a basis for representing hypothetical necessity and possibility, given that the truth of necessity- and possibility-sentences is defined relative to accessible worlds. As suggested in a previous module, accessibility describes which truth-values "matter" in determining whether a sentence is necessarily or possibly true. The accessibility relation itself might best be understood as representing which possibilities matter under a condition that is taken to hold. The condition itself is never expressed in the semantics. It is instead implicit in the arrangement of worlds under the accessibility relation.

Are the K-systems adequate to express hypothetical necessity and possibility? The fundamental strength of K is the property of closure of necessity and possibility over the consequence relation. Basically, closure reflects the fact that truth-values at all worlds are dictated by the same semantical rules. Given the truth of '$\Box(p_1 \wedge p_2)$' at a world, for example, and the consequence that '$p_1 \wedge p_2$' is true at all accessible worlds, we can safely say that 'p_2' is true at those worlds at well, because the semantical rule for '$\wedge$' applies uniformly at all worlds. This result transfers back to give us the truth of '$\Box p_2$' at the original world. So if we want hypothetical modalities to behave uniformly, we would do well to accept the K-systems as a starting-point.

The semantics for early system of C.I. Lewis, *S3*, does not have the closure property.[3] Specifically, we take the simplest case of Closure, Necessitation. It is not the case that if $\vDash_{S3} \Box\varphi$ then $\vDash_{S3} \Box\Box\varphi$. The semantical system *S(3)* is, therefore, not "normal," and its semantics accordingly does not treat truth at worlds in a uniform way. Worlds are segregated into two classes, those that are "normal" and those that are "non-normal." Normal worlds function as in $S(K)$, but at non-normal worlds, sentences of the form $\Diamond\varphi$ are always true, even if φ is false on all models. Similarly, $\Box\varphi$ is always false at a non-normal world, even if φ is true on all models. Thus, Closure is violated in a case where $\Box\varphi$ is true at a (normal) world w but false at an accessible non-normal world w', in which case $\Box\Box\varphi$ is false at w.

Nonetheless, *S3* does come close to closure:

[2] The distinction between what is "absolutely" and "hypothetically" necessary was first made clear by Leibniz.

[3] This applies to the weaker Lewis-systems *S1* and *S2* as well.

If $\models_{S(3)} \varphi \prec \psi$, then $\models_{S(3)} \Box\varphi \prec \Box\psi$,

which is known as "Becker's Rule." Whether such a rule is sufficient for the purposes of formalizing notions of hypothetical possibility and necessity will not be discussed here. At any rate, one would want to use *S3* to represent hypothetical modalities only if one had a reason to invoke conditions under which at some worlds that matter, anything is possible.

We have already noted in connection with the logical modalities that:

$\Box\varphi \not\models_{S(K)} \Diamond\varphi$.

Should we say that what is hypothetically necessary is possible? We can see from $S(K)$ that the only reason the entailment fails is because there are worlds from the point of view of which nothing is possible. Now it may be thought that we need, for some reason, to consider such worlds in our reasoning about possibility. The idea would be that from the standpoint of that world, the condition embodied in the accessibilty relation really cannot hold.

What is odd about this non-consequence is that it seems to say that that a sentence can hold at all worlds under a given hypothesis but not hold at some world under that hypothesis. But this is not what the semantics actually yields. Rather, it can be the case that from the standpoint of a world, there is no way that a sentence can be true under a given hypothesis. But there is no way that it can be false under that hypothesis either, because the condition does not apply to anything accessible. This is what makes it "necessary." And this simply reflects the non-constructive nature of the metalogic. At a world where nothing is hypothetically possible, the condition does not fail to hold with respect to something that is hypothetically possible. So at that world, such a thing has to be regarded as hypothetically necessary.

We have also remarked already that:

$\Box\varphi \not\models_{S(K)} \varphi$.

It may be thought that this consequence should hold in a logic of relative necessity. An argument for this can be made. What this consequence relation means is that whatever is necessary is also true. And surely whatever is necessary under some hypothesis is also true under that hypothesis.

We must be somewhat cautious here, though. What the consequence relation in question actually expresses is that whatever is necessary *at a world* is true *at that same world*. $S(K)$ accommodates the fact that whatever is necessary at a world is true at those worlds that matter *from the standpoint of the home world*, i.e., is true at the accessible worlds. To put the matter in another way, we might wish to understand the hypothesis that governs the accessibility relation as sometimes not holding at a world at which the sentence is evaluated, but holding at all other worlds that meet the condition. For example, we might ponder what is necessary given worlds whose laws of nature are different from those that hold at the actual world.

A third non-consequence in $S(K)$ involves the iteration of the necessity operator:

$\{\Box\varphi\} \not\models_{S(K)} \Box\Box\varphi$.

For hypothetical necessity, this would mean that what must be true under a given condition is such that under that condition, it must be true under the condition. One might wish to deny this principle on the grounds that a metaphysical or physical condition does not have to be a condition of the condition. A physical law, for example, applies to physical phenomena, but it does not say anything about physical laws.

Finally, we have the non-consequence that:

$\{\Diamond\Box\varphi\} \not\models_{S(K)} \Box\varphi$.

The same argument as just given might apply here. It may seem wrong even to think that the laws of nature permit that the laws of nature require something to be the case. More generally, it might be held that with respect to hypothetical necessity and possibility, there should be no principles connecting modal sentences with a single one-place modal operator with those with iterated modal operators. It must be recalled, however, that there are valid formulas with iterated modal operators. Every sentence φ which is valid in $S(K)$ is such that $\Box\Box\varphi$ is valid in $S(K)$, as is $\Box\Box\Box\varphi$, etc.

2 Conditional Logic

The first informal application of $\mathscr{L}$ (indeed, the motivating one) was as a logic of the relation of logical implication. Thus for Lewis, $\varphi \strictif \psi$ was to signify "strict implication," whose truth requires a logical connection between φ and ψ, rather than the merely material connection between them as expressed in sentences of the form $\varphi \supset \psi$. We have seen that in $S(K)$, this kind of sentence is assigned t at a world w if and only if at all worlds accessible to w, if φ has the value t there, then so does ψ.

We saw in the last section that the accessibility relation represents pretty well a notion of hypothetical necessity and possibility. Strict implication, on this reading, amounts to the requirement that there is no world, at which a certain condition holds, where φ is true and ψ is false. What strict implication in the K-systems represents, then, is a localized connection between the values of two sentences. This notion might be of some value. For example, it could reflect the necessity imposed by a law of nature at those worlds at which the law applies. It is still the case that if φ is true at an accessible world w', and $\varphi \strictif \psi$ is true at the home world w, then ψ is true at w'.

The relation of strict implication cannot be said to hold in general, however, given the semantical system $S(K)$. We have already noted the result that $\{\varphi, \varphi \strictif \psi\} \nvDash_{S(K)} \psi$. (It should be noted that in all of Lewis's systems, including his preferred system *S2*, this relation of entailment holds.) So the K-systems are not suitable for the representation of a general relation of logical connection. At best, we have the result that if φ is $S(K)$-valid and $\varphi \strictif \psi$ is true at w, then ψ is true, and indeed valid, at w. Closure of valid sentences over strict implication seems to be a necessary condition for any adequate representation of logical implication. The semantical system $S(K)$, then, at least accommodates this requirement.

It can be argued that aside from being in one respect too weak to represent logical implication, strict implication in the K-systems is too strong, in another respect. This is due to the presence of the two "paradoxes of strict implication," of which Lewis was aware.

Paradoxes of Strict Implication

$$\{\Box\varphi\} \vDash_{S(K)} \psi \strictif \varphi$$
$$\{\sim\Diamond\varphi\} \vDash_{S(K)} \varphi \strictif \psi$$

On any model and at any world at which $\Box\varphi$ is given the value t, $\psi \strictif \varphi$ must also be assigned t. And on any model and at any world at which $\sim\Diamond\varphi$ is given the value t, $\varphi \strictif \psi$ must also be assigned t. A proof of the first "paradox" was given earlier. For the second, suppose that at an arbitrary world w in an arbitrary model M, $\overline{v}(\sim\Diamond\varphi, w) = \mathsf{t}$. So $\overline{v}(\Diamond\varphi, w) = \mathsf{f}$. Then at no world w' accessible to w, $\overline{v}(\varphi, w') = \mathsf{t}$. Hence, at no such world, $\overline{v}(\varphi, w') = \mathsf{t}$ and $\overline{v}(\psi, w') = \mathsf{f}$. Therefore, at every accessible world, if $\overline{v}(\varphi, w') = \mathsf{t}$, then $\overline{v}(\psi, w') = \mathsf{t}$, in which case $\overline{v}(\varphi \strictif \psi, w) = \mathsf{t}$. Since the choice of w and M was arbitrary, the result holds at all worlds on all models, and the entailment is thereby established.

This result is usually interpreted informally as meaning that a necessarily true sentence is strictly implied by any sentence, and that a sentence that is not possibly true strictly implies any sentence.

Critics have argued that no correct formalization of logical implication should permit the result. One line of argument is that in any such implication, the antecedent should be relevant to the consequent. But as can be seen from the form of the two strict implications, as well as from the semantical reasoning, the content of ψ is irrelevant to the truth of the $\mathscr{L}$ sentence-form purporting to represent logical implication.

An equivalent to the sentence-form $\varphi \strictif \psi$ is $\sim(\varphi \circ \sim\psi)$: the truth of φ is inconsistent with the falsehood of ψ. If ψ must be false, then any sentence φ is inconsistent with its truth. Putting the matter this way is reminiscent of a common definition of a valid argument as one in which it is impossible that the premises be true and the conclusion false. This definition is negative, emphasizing the safety of a valid inference. In the worlds of Alan Ross Anderson and Nuel Belnap, two prominent critics of the informal interpretation of strict implication as logical implication, to treat validity this way is to put "safety first."[4]

Another standard definition of validity is that the conclusion must be true given that the premises are true. This looks more positive than the first definition, but the two are treated as equivalent by most logicians. Those who do not think they are equivalent generally interpret the "given that" relation in some constructive way. That is, they think that the conclusion must be constructed from the premises in a certain way. So where φ is necessarily false, there may be no construction of a derivation of ψ from φ. Consider the following derivation-schema.

To prove: $\sim\Diamond\varphi \vdash_{KD} \varphi \strictif \psi$

1	$\sim\Diamond\varphi$	Assumption
2	$\Box$ φ	Assumption
3	$\sim\varphi$	1 SR-$\sim\Diamond$
4	$\sim\psi$	Assumption
5	φ	2 Reiteration
6	$\sim\varphi$	3 Reiteration
7	ψ	4-6 $\sim$ E
8	$\varphi \strictif \psi$	2-7 $\strictif$ I

The assumption of $\sim\psi$ has nothing to do with what transpires in the next two steps, and the derivation of ψ "given that" φ does not really connect φ and ψ at all.

Although in *A Survey of Symbolic Logic* Lewis described the Paradoxes of Strict Implication as "peculiar," he nonetheless embraced them. He gave several arguments to show that an impossible proposition implies any proposition, and a necessary proposition is implied by any proposition. One argument is that an impossible proposition is "absurd" in the sense that it implies its own negation. "The implications of an absurd proposition are no indication of what would be true *if* that absurd proposition were *true*."[5]. What Lewis seems to be arguing is that any implication by an impossible proposition is pefectly harmless, because there is no way in which consequent could be discharged due to the truth of the antecedent. (Strict) Modus Ponens could never be used, because we could not affirm the impossible antecedent.

[4]See *Entailment: The Logic of Relevance and Necessity*, Volume I, pp. 4-5.

[5]*A Survey of Symbolic Logic*, p. 335

A second kind of argument invokes principles of logic which Lewis thought were unimpeachable, and which demonstrate that an impossible proposition implies any proposition. The following is an adaptation of his argument from the *Survey*. First, Lewis invokes two logical principles. The first (Augmentation) is that if $\varphi \prec \psi$, then $(\varphi \wedge \gamma) \prec \psi$. The second (Antilogism) is that if $(\varphi \wedge \gamma) \prec \psi$, then $(\varphi \wedge {\sim}\psi) \prec {\sim}\gamma$. From these two principles it follows straightforwardly that if $\varphi \prec \psi$, then $(\varphi \wedge {\sim}\psi) \prec {\sim}\gamma$.

Now suppose that φ is impossible, in which case it implies its own negation. Then we have $\varphi \prec {\sim}\varphi$. By the result just given, $(\varphi \wedge {\sim}{\sim}\varphi) \prec \gamma$. Since ${\sim}{\sim}\varphi$ is equivalent to φ, we have $(\varphi \wedge \varphi) \prec \gamma$, which is equivalent to saying that $\varphi \prec \gamma$. He then concluded, "In this respect, then in which the laws of Strict Implication seemed possibly not in accord with the 'proper' sense of 'implies', we have demonstarated that they are, in fact, required by obviously sound logical principles, though in ways which it is easy to overlook."[6]

Lewis and Langford advanced a similar type of argument in terms of a derivation that apparently shows that 'ψ' is derivable from the impossible '$\varphi \wedge {\sim}\varphi$' by intuitively sound principles.[7]

To prove: $\varphi \wedge {\sim}\varphi \vdash_{SD} \psi$

1	$\varphi \wedge {\sim}\varphi$	Assumption
2	φ	1 $\wedge$ E
3	${\sim}\varphi$	1 $\wedge$ E
4	$\varphi \vee \psi$	2 $\vee$ I
5	ψ	3 4 Disjunctive Syllogism

Which step are we to give up if we are to deny that ψ is logically implied by $\varphi \wedge {\sim}\varphi$? Each one seems to be a legitimate logical implication in itself. It is hard to see how one could give up $\wedge$ Elimination, so it appears that the choice comes down to Disjunctive Syllogism and $\vee$ Introduction. "Relevance logics" give up the former (as well as Antilogism) and "connectivist" or "analytical entailment" logics give up the latter (as well as Augmentation).[8]

Despite Lewis's argumentation in favor of accepting the Paradoxes, there seems to be a conflict between them and what he advocated in his original article on modal logic. As has already been shown, $\varphi \prec \psi$ is equivalent to $\Box(\varphi \supset \psi)$. This is in turn equivalent to $\Box({\sim}\varphi \vee \psi)$, which is a case of intensional disjunction. Lewis had claimed that the truth of such a disjunction can be known without the truth-values of the disjuncts being known. "Its truth can be known, while it is still problematic *which* of its lemmas is the true one. It has a truth which is prior to the determination of the facts in question."[9]

But in a case such as '$\Box p_1 \vee (p_2 \supset p_1)$),' the truth-value of the disjunction is known through that of the right disjunct '$p_1 \supset p_1$' alone. Since 'p_1' is a formula, the value of the disjunction cannot be known from the logical form of 'p_2,' which permits either the value t or the value f. We must know that '$p_1 \supset p_1$' is true in order to know that the disjunction is true. Ironically, this same consideration seems to apply to Lewis's own example of a sentence that fails the test for intensional disjunction, "Today is Monday or $2 + 2 = 4$" (given that the right disjunct is taken to be a necessary truth).

[6] *A Survey of Symbolic Logic*, pp. 338-339.
[7] Lewis and Langford, *Symbolic Logic*, p. 250.
[8] Australian logicians prefer the expression 'relevant logics.'
[9] "Implication and the Algebra of Logic," p. 524.

A way to avoid the paradoxes which is in the spirit of Lewis's first paper is to give up the use of Disjunctive Syllogism on the result of the use of $\vee$ Introduction. If we use $\vee$ Introduction, we already take it that one of the disjuncts is true. In the present case, we take 'p_1' to be true. So the disjunction is not intensional, by Lewis's original criterion. It is not a matter of the logical relation between 'p_1' and 'p_2' that the disjunction holds. Now given that we already take 'p_1' to be true, it could be argued, the only legitimate use of Disjunctive Syllogism would be to re-assert the truth of 'p_1', given the falsehood of 'p_2' (which, of course, we do not have).

Consider Lewis's example, "Either the today is Monday or $2 + 2 = 4$." Suppose I assert this confidently solely on the grounds that "$2 + 2 = 4$" is a necessary truth. But then suppose further that I come upon the information that I have been deluded by a powerful evil deceiver into believing this, and in fact the sum of 2 and 2 is not 4.[10] Given the reasoning of the preceding argument, I would be entitled to conclude, "Well, then, at least I know that today is Monday!" This is, of course, an absurd conclusion to draw. What I would do instead is to *take back* my assertion of the disjunction. So the idea is that in the presence of the negation of a sentence used to establish a disjunction by $\vee$ Introduction, the disjunction should not be used for further ends, but instead should be withdrawn.[11] This suggestion will not be developed in any more detail here. I only note that this criticism treats logic as providing rules of good inference.

The three responses to Lewis's arguments have in common the requirement that changes be made to the underlying non-modal Propositional Logic. This indicates the close relation between strict implication and the material conditional of which it is the necessitation. For better or worse, truth-functional Propositional Logic is the basis for the modal systems to be treated here. So we shall have to be content to assess the adequacy of strict implication without changing the underlying logic.

It should be noted that the expression of logical implication is only one of the several roles of conditionals in natural language. An important class of conditionals is the counterfactual, or contrary-to-fact conditional. David Lewis and Robert Stalnaker have shown how to use possible-worlds semantics to interpret counterfactuals (though their systems differ). These semantical systems add an additional relation of *closeness* or *nearness* of accessible worlds. It would be appropriate to explore such systems here, but time constraints prohibit this.[12]

3 Deontic Logic

It does not seem controversial that people make inferences with premises involving obligation and permission. I might, for example, take as a premise that it is obligatory that I pay federal income taxes and state income taxes, and conclude it is obligatory that I pay federal income taxes. This appears to be a valid inference, which would have a place in a logic of obligation, or *deontic logic*.

Deontic logic could be couched in the syntax of alethic modal logic with its '$\Box$' and '$\Diamond$' operators. But it is customary with this (and the other variants of modal logic that follow) to replace them with operators which are more suggestive of their intended interpretation. This produces a modal language distinct from $\mathscr{L}$, but which parallels it strictly. The semantical and derivational systems defined on the specialized modal language are different as well, as the semantical and derivational rules apply to operators other than the '$\Box$' and the '$\Diamond$.' To keep matters straight, we will give special names to the specialized systems.

[10]The possibility of which is suggested in Descartes's First Meditation.
[11]Broadly speaking, such a system would not be monotonic.
[12]Bonevac's *Deduction* contains a good exposition.

In deontic logic, sentences of $\mathscr{L}$ whose main operator is the '$\Box$' could then be considered analogous to sentences which indicate obligation for an agent. Sentences of $\mathscr{L}$ whose main operator is the '$\Diamond$' would be analogous to sentences indicating permissibility for an agent. It is common practice in deontic logic to use an obligation operator 'O' and a permission operator 'P.' The semantical and derivational systems for the resulting deontic modal language will be called *DKI* and *DKD*, respectively.

One notable property of the *DK*-systems is duality. What is permitted (for an agent at a time in a situation) is what the agent is not obligated not to do. If I am permitted to pay my federal income tax by check, I am under no obligation not to do so. Similarly, what is obligatory is not permitted not to be done. If I am obligated to pay federal income tax, I am not permitted not to pay it. It seems that these two versions of duality correctly express the way we talk about obligation and permission.

Closure appears to be a feature of the way we reason about obligation and permission. It seems obviously to be true that my paying my tax follows from the fact that I file my federal income tax return and pay my tax. By closure, given that I am obligated to file my federal income tax return and to pay my tax, it follows that I am obligated to pay my tax. We must be careful in understanding closure for permission. For example, it follows from the fact that I pay my income tax with a check that either I pay my income tax with a check or I pay my income tax with play money. Given closure, and the fact that it is permissible to pay my income tax with a check, it follows that it is permissible that either I pay my income tax with a check or I pay my income tax with play money.

This result might seem counter-intuitive, but it is really just odd-sounding. For it does not follow, in the *DK*-systems, from the fact that it is permissible that I pay my income tax with a check or I pay my income tax with play money that it is permissible that I pay my income tax with play money, which would be a counter-intuitive result. In the semantical system, $\mathrm{D}(\varphi \vee \psi) \nvDash_{DKI} \mathrm{D}\psi$.

The *KD*-systems are clearly too weak to yield a satisfying deontic logic. It seems intuitively right to say that what is obligatory is permissible, i.e., that $\{\mathrm{O}\varphi\}$ entails $\mathrm{P}\varphi$. If I must file my federal income tax return, then I am allowed to do so. But here we run up against the fact that this implication does not hold in the *DK*-systems.

Having considered the suitability of the *DK*-systems to capture our reasoning about obligation and permission, we can ask how to understand the accessibility relation in the semantical system $S(K)$. On an analogy with hypothetical necessity, we can think of accessibility in the context of deontic logic as reflecting the conditions for right behavior that hold at a world—perhaps a moral law holding for all cognitive agents, or some civil law applying to all residents of a nation.

On that view, worlds w' accessible to w can be thought of as worlds which are subject to to the law that holds at w. For example, I live in a world where I am governed by the United States Tax Code. I am required to file a federal income tax return if I have income, and I am permitted to file it electronically and permitted to file it on paper. In the semantics, this would be represented as that in every world compatible with that law, I file a federal income tax return if I have income. In some such worlds, I file my taxes electronically, and in others I file them on paper.

Daniel Bonevac has a nice way to look at deontic accessible worlds is in terms of ideals that hold at worlds.

> We may imagine each possible world as looking to other possible worlds for its moral or practical values. These worlds need not be prefect, but they do need to be ideal in some respects. In particular, the worlds that our world holds up as ideal should be ones in which all the obligations that hold in our world are fulfilled. We can imagine ourselves as thinking of how things ought to be and measuring the current state of affairs, or our own behavior, by that yardstick. But not all worlds need have the same ideals. We can

perhaps imagine that some worlds are better than ours, in that none of our obligations go unfilfilled there. Perhaps the moral standards of such a world, however, are so high that new obligations arise in that world. In contrast, there may be worlds so destitute that they hold up our world as an ideal. (*Deduction*, p. 313)

If we buy into this picture, then we will want our system of deontic logic to be relatively weak. If an accessible world has higher standards than our own, the fact that those standards hold there does not affect the standards we have in the present world. In that case, the accessibility relation should not be transitive.

Although the standard systems of deontic logic, the *D*-systems of the next module, are stronger than the *K*-systems, there is one reason to claim that the *K*-systems are in a sense too strong. Consider the $S(K)$ entailment $\{\Box\varphi, \Box\psi\} \vDash_{S(K)} \Box(\varphi \wedge \psi)$. On a deontic interpretation, this means that if two actions are individually obligatory, then both actions together are obligatory. If this is the case, then conflicts of obligation are ruled out, which, to some, might seem undesirable. To avoid this consequence, it is required that closure under conjunction be given up, in which case, the resulting system would not be "normal" in the sense defined in the last module. Axiomatic and rule-based systems, with their interpetations by way of "neighborhood semantics," have been offered in the literature but will not be pursued here.

4 Doxastic Logic

The development of a logic of belief, or *doxastic logic*, was initiated by Jaakko Hintikka, one of the originators of possible-worlds semantics.[13] A doxastic logic would indicate what follows from what one believes. We will specify a syntax for doxastic logic and *BK*-systems (doxastic *K*-systems) analogous to the *K*-systems.[14] As with deontic logic, the semantical and derivational rules for the doxastic modal operators in the *BK* systems are strictly analogous to those for the respective *K*-systems.

We begin by replacing the '$\Box$' operator of $\mathscr{L}$ with an operator indicating that a subject believes something at a time. So there will be an operator '$\mathrm{B}_{s,t}$' with a variable index '*s*' for the subject and a variable index '*t*' for the time. The operator replacing the '$\Diamond$' operator, '$\mathrm{P}_{s,t}$,' is somewhat artificial. Given duality, $\mathrm{P}_{s,t}\varphi$ says that at time *t*, *s* does not believe that φ is false. To put it another way, φ's being true at *t* is compatible with what *s* believes at the time. I might say, "It could be that I will live to be 100 years old," meaning that nothing I believe is inconsistent with my living to be 100.[15]

There is some reason to think that we make inferences from what is believed. For example, from the premise that I believe that Newton was Master of the Mint and a member of the Royal Society, you might conclude that I believe that Newton was Master of the Mint. However, the converse inference may not be valid, as it is in the *BK*-systems. Suppose I believe that Guadalajara is in Mexico and I believe that global warming is occurring. Does it follow that I believe the conjunction, that Guadalajara is in Mexico and global warming is occurring? I may not have put the two separate beliefs together.

Or take another case. When I was a child, I had never heard of tectonic plate theory, since nothing about it had been published at the time. I believed at one time then that I was eating an apple. Did I thereby believe then that I was eating an apple or earthquakes are caused by the motion of tectonic plates? The disjunction follows from the single sentence, and so by closure,

[13] See his classic *Knowledge and Belief: The Logic of the Two Notions*.

[14] We have already used the letter 'D' for the denotic systems, so we instead use 'B' for 'belief.'

[15] This is a doxastic, not an alethic, 'could.'

my belief in the disjunction should follow from my belief in the single sentence. More generally, a subject will at any time believe all the consequences in the *BK*-system of what the subject believes at that time.

The limiting case is where a sentence φ is valid in $S(BK)$. It follows from the rules of $S(BK)$ that the subject believes φ. The notion of its being valid that someone believes something, without any constraints on that subject's even having considered it, is at best an idealization. So what the *BK*-systems yield is a logic of the beliefs of a subject that is a kind of "logical saint" in the sense that it believes all the logical truths and what follows from them. Clearly, this is an idealized notion of belief. It might be thought of as a kind of deontic notion, a notion of what one *should* believe if one were ideally rational. The subject should believe what holds as a matter of the logic and what follows in the logic from what the subject believes.

If we view doxastic logic in this light, we can ask what to make of the accessibility relation. A sentence $\mathrm{B}_{s,t}\varphi$ has the value t at a world w just in case φ has the value t at all worlds accessible to w. We have been thinking of these worlds as embodying a condition that is assumed to hold. We will work with the condition that the accessible worlds are those worlds which for the subject are "live" possibilities, i.e., those which are not ruled out on the basis of the subject's beliefs. Accessibility does nothing to explain what belief is, but is only concerned with the consequences of what one believes.

Given duality, we have the result that $\{\sim\mathrm{P}_{s,t}\sim\varphi\} \vDash_{S(BK)} \mathrm{B}_{s,t}\varphi$, i.e., that whatever is not incompatible with what one believes is believed. This result is initially quite implausible. But consider the extreme case, where an model contains a dead end, a world to which no other world is accessible. The condition $\sim\mathrm{P}_{s,t}\sim\varphi$ holds at any such world, and so $\mathrm{B}_{s,t}\varphi$ also holds there. A dead end signifies that there is nothing that is ruled out by one's beliefs, and so everything is believed.

This result seems implausible because it seems that the truth of $\sim\mathrm{P}_{s,t}\sim\varphi$ sets up a condition which requires a belief that φ at that world. However, we must keep in mind that the accessible worlds are themselves defined by what the person believes. So the truth of $\sim\mathrm{P}_{s,t}\sim\varphi$ at a dead-end is a reflection of the condition that the subject believes what he does, rather than being itself a condition that determines what the subject believes.

Now of course a logical saint is not the sort of subject that would believe everything, even what is inconsistent. So it seems desirable that the semantics for doxastic logic require that each world have at least one world accessible to it. In that way, whatever is false at that world is not believed by the subject. And in general, the subject will not be represented as believing anything inconsistent, since no world allows for any sentence and its negation to receive both of the values t and f.

We would not, however, want to hold that the truth of $\mathrm{B}_{s,t}\varphi$ semantically entails the truth of φ. What we believe need not be true. So in this respect, at least, $S(BK)$ is not too strong for the purposes of doxastic logic.

Some hold that what one believes, one believes one believes. If this condition is desired, then the accessibility relation will have to be transitive, as in the semantical system for *S4*. On the other hand, we should not require that each world be accessible to itself. This is tantamount to holding that what one believes is true. Even the "logical saint" is not infallible in factual matters, though he is in matters of logic.

Finally, it is unclear whether we should hold that if it is compabtible with what a person believes that the person believes something, then the person believes it. This topic will be discussed further in Module 12.

5 Epistemic Logic

On the model of doxastic logic, Hintikka used an analogue of the '□' operator, '$K_{s,t}$', to indicate a subject's knowing something at a time. The "epistemic possibility" operator '$F_{s,t}$' indicates compatibility with what one knows. We might wish to make inferences with knowledge-sentences as premises and conclusion, and hence to develop an *epistemic logic*. Systems of epistemic logic analogous to the *K*-systems will be called *KK*-systems.

Another similarity between doxastic and epistemic logic is the way they treat the accessibility relation. Accessible worlds are those which are not ruled out by what one knows. I might say that it could rain tomorrow. This is consistent with what I know about current conditions, the weather forecast, the typical weather-patterns for this time of the year, etc. What was said about duality in doxastic logic also applies here. If it is not the case, for all I know, that $\sim\varphi$, then I know that φ. As with doxastic logic, we will want to move to systems stronger than *KK*-systems to avoid the trivial truth of knowledge-sentences which is allowed by the presence of dead-end worlds in an model.

Given Closure in the *KK*-systems, if a sentence φ is entailed by a set of sentences Γ, then if one knows all the sentences in Γ, then one knows φ. For example, the sentence "This is not a mule" follows from the sentences "I now see that this is a zebra" and "If this is a zebra, then this is not a mule." Given Closure, we have it that from "I know now that this is a zebra" and "I know now that if this is a zebra, then this is not a mule," it follows that, "I know now that this is not a mule." Symbolically, we would have $\{K_{i,n}Z, K_{i,n}(Z \supset \sim M)\} \vDash_{S(KK)} K_{i,n} \sim M$.[16]

Some philosophers have attacked the closure condition as leading to skepticism.[17] For example, let 'H' symbolize 'I am a victim of an elaborate hoax in which a mule is painted to look like a zebra.' We might then have the following instance of closure: $\{K_{i,n}Z, K_{i,n}(Z \supset \sim H)\} \vDash_{S(KK)} K_{i,n} \sim H$. It follows from the fact that I know now that I see a zebra and that if I see a zebra, then I am not a victim of an elaborate hoax, that I know now that I am not a victim of an elaborate hoax.

But, the critics claim, I do not know that I am not the victim of an elaborate hoax, because I cannot exclude the possibility of such deception given the knowledge that I have: $\sim K_{i,n} \sim H$. So from the standpoint of a world w, there is an accessible world $\mathbf{w}_i$ at which 'H' is not assigned t, and so at w, $\sim K_{i,n}Z$ is assigned t. Perhaps the best response to this criticism is to fall back on the fact that we are working with a very idealized concept of knowledge, and that the logical saint would be obliged to exclude the possibility of every kind of deception.

The *KK*-systems appear to be too weak as logics of knowledge. One reason is that it seems that it should follow from the fact that someone knows that φ that φ is epistemically possible, i.e., is compatible with what the person knows. However, we have the result that $\{K_{s,t}\varphi\} \nvDash_{S(KK)} P\varphi$. It seems wrong to allow knowledge in the case where no world is accessible to a given world, and thus there is no world containing information about what is compatible with what one knows.

We will want to say that what is known is true. But we have the result that $\{K_{s,t}\varphi\} \nvDash_{S(KK)} \varphi$. So systems stronger than the *K*-systems are needed. Some epistemologists think that if one knows that φ, then one knows that one knows that φ. Representing this requires a stronger system as well.

Many theorists of knowledge accept the "K-K" thesis, according to which if a person knows that φ, then that person knows that he knows that φ. This would be a feature of a system of epistemic logic based on *S4* and its equivalents.

Unlike the analogous case in doxastic logic, it seems clear that the fact that it is epistemically possible that one knows something does not imply that one knows it. Suppose that it is compatible

[16]This famous example is taken from Fred Dretske's paper, "Epistemic Operators." It will be developed in what follows.

[17]See, for example, the articles reprinted in Part Three of *Skepticism: A Contemporary Reader*, by Keith DeRose and Ted A. Warfield.

with what I know that there is a bird on the roof of my house right now. This fact is not precluded by anything known by me. Surely I do not thereby know that there is a bird on my roof. Unlike belief, which can be thought of as a matter internal to myself, knowledge requires the truth of what is known, and this is something that does not depend entirely on me.

6 Temporal Logic

The modern logic of time, *temporal logic*, has its origins in the work of Arthur Prior.[18] Times are quite naturally adapted to modal interpertation. The ancient Megarian philosopher Diodorus Cronus interpreted necessity and possibility themselves temporally, according to Cicero.

> Diodorus defines the possible as that which either is or will be, the impossible as that which, being false, will not be true, the necessary as that which, being true, will not be false, and the non-necessary as that which either is already or will be false. (Cited in Kneale and Kneale, *The Development of Logic*, p. 117).

The interpretation of the alethic modalities as temporal is known as the *Diodorian* interpretation. It can be found in early modern philosophers such as Hobbes.[19] In what follows, we shall not be working with the Diodorian interpretation of alethic modalities, but rather with an interpretation of the modal operators simply as standing for times.

6.1 Temporal Interpretation of Accessibility

We ordinarily think of time as whole as composed of discrete units that may be called "times" or perhaps "moments." Thought of in the most general way, these times are ordered by a directional relation of precedence. If one time is "before" another, then the second time is "after" the first. We may wish to add to this generalized conception of time. For example, we may think that there is a first time, a last time, both, or neither. Further, we may think of time as having a linear structure, versus, say, a branching structure.

The "possible worlds" in a frame can accordingly be interpreted as times and accessibility can be interpreted as reflecting the relation between times.[20] Because in $S(K)$ there are no resrictions on accessibility, many setups are possible. There might be only one time. Or there might be times that are not ordered by the before/after relation. This is sufficient for a minimal logic of temporal operators.

Let us say that if one time is accessible to another, that time is later than the other.[21] If Rww'. then w' is later than w. We can say dervatively that w' is earlier than w if and only if w is later than w', i.e., $Rw'w$. This will allow us to use a single accessibility relation to generate semantics for sentences governed by future-looking and by past-looking temporal operators.

6.2 Future-Looking Operators

Given our choice to take Rww' to mean that w' is later than w, the analogues of the '$\Box$' and '$\Diamond$' operators will be temporal operators that point toward the future, in that future times are what

[18] See *Time and Modality* and *Past, Present and Future*.

[19] *Concerning Body*, 1656, Part II, Chapter X "Of Power and Act."

[20] The "worlds" in a frame can also be taken as time-slices of the universe, as moments, or in some other way that coheres with one's view of the nature of time and its relation to the world.

[21] Alternative, we could have said that the accessible time is the earlier time.

are accessible to a given time.[22] In the literature, the two basic future-looking operators are the following, on the analogy with the '$\Box$' and '$\Diamond$' operators, respectively:

Gφ: it will always be the case that φ.
Fφ: it will at some time be the case that φ.

We can give the expected semantical rules for these operators, on the analogy of the pairs **SR**-$\Box$/**SR**-$\Box$ and **SR**-$\Diamond$/**SR**-$\Diamond$:

SR-G $\bar{v}(\mathrm{G}\varphi, w) = \mathrm{t} \leftrightarrow (\Pi w')(Rww' \to \bar{v}(\varphi, w') = \mathrm{t})$.

SR-F $\bar{v}(\mathrm{F}\varphi, w) = \mathrm{t} \leftrightarrow (\Sigma w')(Rww' \wedge \bar{v}(\varphi, w') = \mathrm{t})$.

The semantical rules work just as in $S(K)$, the only difference being in symbols used as modal operators. "It will always be the case that φ" is true at a time if and only if φ is true at all future times. "It will at some time be the case that φ" is true at a time just in case φ is true at some future time.

6.3 Past-Looking Operators

Corresponding to the future-looking operators are two past-looking operators:

Hφ: it has always been the case that φ.

Pφ: it has at some time been the case that φ.

For it always to have been the case that φ relative to a given time w, it must be that at each time before w, φ is the case. As we saw, this is to say that w is later than each of the earlier times. So we can use the converse of accessibility to express the truth-condition for the 'H' and 'P' operators.

SR-H $\bar{v}(\mathrm{H}\varphi, w) = \mathrm{t} \leftrightarrow (\Pi w')(Rw'w \to \bar{v}(\varphi, w') = \mathrm{t})$.

SR-P $\bar{v}(\mathrm{P}\varphi, w) = \mathrm{t} \leftrightarrow (\Sigma w')(Rw'w \wedge \bar{v}(\varphi, w') = \mathrm{t})$.

"It has always been the case that φ" is true at a time if and only if φ is true at all the times than which φ is later. "It will at some time be the case that φ" is true at a time just in case φ is true at some time than which φ is later. Less awkwardly expressed, the conditions are that φ is true at all times before the given time, and that φ is true at some time before the given time.

Adding the four rules to the semantical system $S(0)$ for Propositional Logic, we get a semantical system which we will call $S(K_t)$, after the well-known the axiomatic system $A(K_t)$.[23] The latter has been called a "minimal" or "basic" tense logic.

6.4 The Axiom System $A(K_t)$

Before turning to $S(K_t)$, we will display the axiom system $A(K_t)$. This system takes the 'G' and 'H' operators to be primitive and defines the 'F' and 'P' operators on the analogy with duality for the '$\Box$' and '$\Diamond$' operators. To the axioms for non-modal sentential logic, we add four new axioms:

$\vdash_{A(K_t)} \mathrm{G}(\varphi \supset \psi) \supset (\mathrm{G}\varphi \supset \mathrm{G}\psi)$,

$\vdash_{A(K_t)} \mathrm{H}(\varphi \supset \psi) \supset (\mathrm{H}\varphi \supset \mathrm{H}\psi)$,

$\vdash_{A(K_t)} \varphi \supset \mathrm{GP}\varphi$,

[22] Had we made the other choice, then we would have begun with operators that point toward the past.

[23] Here we follow historical nomenclature. It would be inappropriate to call this system '$A(TK)$,' as it contains more modal operators than does K and hence is an extension of K.

$\vdash_{A(K_t)} \varphi \supset \mathrm{HF}\varphi$.

To this are added two "Necessitation" rules:

From $\vdash_{A(K_t)} \varphi$ to infer $\vdash_{A(K_t)} \mathrm{G}\varphi$,

From $\vdash_{A(K_t)} \varphi$ to infer $\vdash_{A(K_t)} \mathrm{H}\varphi$.

6.5 The Semantical System $S(K_t)$

Because the semantical rules for the 'G' and 'H' operators are based directly on the definitions in $S(K)$, we can assert that the two closure principles hold:

$\{\gamma_1 \ldots \gamma_n\} \vDash_{S(K_t)} \varphi \to \{\mathrm{G}\gamma_1 \ldots \mathrm{G}\gamma_n\} \vDash_{K_t I} \mathrm{G}\varphi$, and

$\{\gamma_1 \ldots \gamma_n\} \vDash_{S(K_t)} \varphi \to \{\mathrm{H}\gamma_1 \ldots \mathrm{H}\gamma_n\} \vDash_{K_t I} \mathrm{H}\varphi$, and

If φ follows from a set of sentences, then if what is expressed by those sentences will always be the case, then φ will always be the case. If φ follows from a set of sentences, then if what is expressed by those sentences has always been the case, then φ has always been the case.

We will prove the first closure results using a metalogical derivation.

Semantical proof of closure for the 'H' operator

1	$\{\gamma_1 \ldots \gamma_n\} \vDash_{K_t I} \varphi$	Assumption
2	$(\overline{v}(\gamma_1, w) = \mathrm{t} \wedge \ldots \wedge \overline{v}(\gamma_n, w) = \mathrm{t}) \to \overline{v}(\varphi, w) = \mathrm{t}$	1 Df. '$\vDash$'
3	$\quad \overline{v}(\mathrm{H}\gamma_1, w) = \mathrm{t} \wedge \ldots \wedge \overline{v}(\mathrm{H}\gamma_n, w) = \mathrm{t}$	Assumption
4	$\quad\quad Rw'w$	Assumption
5	$\quad\quad \overline{v}(\gamma_1, w') = \mathrm{t} \wedge \ldots \wedge \overline{v}(\gamma_n, w') = \mathrm{t}$	3 **SR**-H
6	$\quad\quad \mathbf{v}(\varphi, w') = \mathrm{t}$	2 5$\to$ E
7	$\quad Rw'w \to \mathbf{v}(\varphi, w') = \mathrm{t}$	4-6 $\to$ I
8	$\quad \overline{v}(\mathrm{H}\varphi, w) = \mathrm{t}$	7 **SR**-H
9	$\overline{v}(\mathrm{H}\gamma_1, w) = \mathrm{t} \wedge \ldots \wedge \overline{v}(\mathrm{H}\gamma_n, w) = \mathrm{t} \to \overline{v}(\mathrm{H}\varphi, w) = \mathrm{t}$	2-8 $\to$ I
10	$\{\mathrm{H}\gamma_1 \ldots \mathrm{H}\gamma_n\} \vDash_{K_t I} \mathrm{H}\varphi$	9 Df. '$\vDash$'

Exercise. Prove that closure holds for the 'G' operator.

An interesting semantical result involves sequences of two operators, one future-looking and one past-looking: $\mathrm{GP}\varphi$ and $\mathrm{HF}\varphi$. Specifically, we have the following two results, which echo the third and fourth axioms of $A(K_t)$:

$\{\varphi\} \vDash_{S(K_t)} \mathrm{GP}\varphi$,

$\{\varphi\} \vDash_{S(K_t)} \mathrm{HF}\varphi$.

If φ is true at a time, then at every future time there is some past time at which it was true, and at every past time there is some future time at which it is true. The time at which φ is true serves as the "some" time which is required to make the modal sentences true. This can be seen from the following proof-skeleton for the first semantical entailment.

Semantical proof that $\varphi \vDash_{K_t I} \mathrm{GP}\varphi$

1	$\overline{v}(\varphi, w) = \mathrm{t}$	Assumption
2	Rww'	Assumption
3	$Rww' \wedge \overline{v}(\varphi, w) = \mathrm{t}$	1 2 $\wedge$ I
4	$(\Sigma w')(Rww' \wedge \overline{v}(\varphi, w) = \mathrm{t})$	3 Σ I
5	$\overline{v}(\mathrm{P}\varphi, \mathrm{w}') = \mathrm{t}$	4 **SR**-P
6	$Rww' \rightarrow \overline{v}(\mathrm{P}\varphi, \mathrm{w}') = \mathrm{t}$	2-5 $\rightarrow$ I
7	$\overline{v}(\mathrm{GP}\varphi, \mathrm{w}) = \mathrm{t}$	6 **SR**-G

Exercise. Prove that the other semantical entailment holds.

We can diagram the above proof by keeping in mind that the accessibility arrow can be read "backward." If one time is later than another, then the former is earlier than the latter. With this in mind, we can assign a value to Pφ at a world by looking back at the world to which it is accessible to discover whether φ is true there. This gives the following result.

w_1	$\longrightarrow$	w_2
φ		
t		
		Fφ
		t
GFφ		
t		

We have called the system $S(K_t)$ a minimal system of temporal logic. The system is indeed minimal with respect to a logic of time. There are a number of other constraints we undoubtedly would like to impose on the semantics. For example, it seems obvious that no time is later (or earlier) than itself. But there are frames in which Rww. It also seems reasonable to demand that each time be either earlier or later than any other time. And we should also want in our semantics a restriction that no time is both earlier and later than another time. More generally, we would like the accessibility relation to generate a linear ordering of times. We may also wish to represent a first time or a last time. All this requires restrictions on the R relation. Such restrictions will be discussed in later modules.

6.6 A Note on Derivations

Corresponding to the specialized semantical systems we have been discussing are specialized derivational systems. The rules for these systems are the same as those for the alethic system *KD*, except for the symbols involved. With respect to strict scope lines, the appropriate analogue of the '□' should be written to the left of the line.

The case of K_t requires that new rules be given. First, we need to have two separate kinds of restricted scope lines, one each for the 'G' and 'H' operators. Second, we need rules of inference to reflect the third and fourth axioms, which mix future-looking and past-looking operators. These are easily provided with special versions of Strict Reiteration.

Strict Reiteration for 'G' and 'P'

$$
\begin{array}{l|l}
& \varphi \qquad\qquad \text{Already Derived} \\
\text{G} & \quad \vdots \\
& \quad \mathrm{P}\varphi \qquad \text{SR-G-P}
\end{array}
$$

Provided that Pφ is strictly reiterated across exactly one restricted scope line.

Strict Reiteration for 'H' and 'F'

$$
\begin{array}{l|l}
& \varphi \qquad\qquad \text{Already Derived} \\
\text{H} & \quad \vdots \\
& \quad \mathrm{F}\varphi \qquad \text{SR-H-F}
\end{array}
$$

Provided that Fφ is strictly reiterated across exactly one restricted scope line.

Proof of the axioms is obvious and is left to the reader as an exercise. Inspection of the earlier semantical proof that $\{\varphi\} \vDash_{K_i I}$ GPφ reveals the similarity between semantical reasoning and the derivation rule for 'G' and 'P.'