Module 15
Introduction to Modal Predicate Logic

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Contents

1 The Dimensions of Modal Predicate Logic

In the previous two modules, two distinct languages of predicate logic were presented, classical Predicate Logic *PL* and free predicate logic *FPL*. Both languages can be extended by the addition of modal operators. Extensions of *PL* (the "*Q1-x*" systems) will be treated in the final module. In the present module, the language Modal Predicate Logic (*MPL*) is an extension of the language Free Predicate Logic (*FPL*). Its syntax is generated by adding modal operators to the syntax of *FPL*. This will allow such sentences as '$(\exists x)\Box Fx$,' in which a modal operator occurs in the scope of a quantifier, and '$\Box(\exists x)Fx$' where a quantifier lies in the scope of a modal operator. Semantical systems for *MPL* vary a great deal in their treatment of sentences of this kind.[1]

1.1 Modality *De Re* and *De Dicto*

Modal Predicate Logic allows for the representation of modality "*de re*" and modality "*de dicto*." In general, a *de re* modality is one in which a quantifier has a modal operator in its scope. If the operator is alethic, then its form is either $(\forall x)\Box\alpha(x/u)$, $(\forall x)\Diamond\alpha(x/u)$, $(\exists x)\Box\alpha(x/u)$, or $(\exists x)\Diamond\alpha(x/u)$.[2] A *de dicto modality*, by

[1] For a systematic introduction to systems of Modal Predicate Logic, see James Garson, "Quantification in Modal Logic", in Gabbay and Guenthner, eds., *Handbook of Philosophical Logic*, Volume II, pp, 249-307.

[2] One could also list schemata for the other modal operators which correspond to '$\Box$' and '$\Diamond$.'

contrast, is one in which a quantifier is in the scope of a modal operator. For alethic modalities, its form is either $\Box(\forall x)\alpha(x/u)$, $\Diamond(\forall x)\alpha(x/u)$, $\Box(\exists x)\alpha(x/u)$, or $\Diamond(\exists x)\alpha(x/u)$.

One kind of *de re* alethic modality predicates something necessarily of an object. Suppose we are talking about positive integers, and we want to say that there is a number that necessarily is odd, where the property of being odd is symbolized by the predicate letter 'F.' Then we can write: '$(\exists x)\Box Fx$.' There are *de dicto* modalities by which one asserts that necessarily, there is something of which the predicate holds. In our example of positive integers, we might wish to say that necessarily, there is a number that is odd, which would be symbolized as '$\Box(\exists x)Fx$.' One of the interesting questions in the logic of quantifiers and modal operators is whether modalities *de re* and *de dicto* imply one another. It will turn out that the choice of systems of *MPL* will bear directly on this question.

In some applications, there are inferences that we would want to block. In doxastic logic, for example, we would not want to be able to conclude a *de re* modality from a *de dicto* modality. If someone believes that there exists a twenty-foot-tall human being, it ought not to follow that there is a twenty-foot-tall human being of whom the person believes to exist. So the choice of a system of *MPL* that handles these modalities properly is as important as the choice of a system of *MSL* that blocks the inference from one's believing something to its being true.

1.2 Semantical Systems for Modal Predicate Logic

There are many aspects of the semantics for Predicate Logic and for modal logic which allow for variation when they are combined.[3] In the first place, there are many systems of Modal Sentential Logic on which Modal Predicate Logic can be based. Secondly, the semantics for modal and predicate logics have complicating features not found in Sentential Logic. The two most important are these. Modal semantics has as its foundation frames containing sets of possible worlds at which sentences are assigned values by a valuation function. Semantics for Predicate Logic is based further on a non-empty domain of discourse, which contains the individuals which are the referents of names and serve as the values of variables.

When the two are combined, there are several ways in which the possible worlds can be related to the domain of discourse. The simplest relation is that in which there is a single domain that applies across all possible worlds. This means that each world is populated by exactly the same objects. However, much of the appeal of modal logic lies in the idea that we can use it to represent states of affairs which are not actual. One way in which things might be different is that different objects exist. To capture this, one might wish to allow that each possible world has its own domain of discourse. This complicates the semantics, as will be seen.

Another issue is how to treat the assignment of constants to members of the domain of discourse (at a world, if the domain varies across worlds). Some semantical systems require that constants refer to exactly the same individuals, no matter what the world at which sentences containing them are evaluated. Or if domains vary, the constants designate the same thing in all those worlds at which the designee exists.[4] Following Kripke we will call constants, and more generally terms, which designate the same member of the domain when they designate at all "rigid designators." Other systems might allow that a constant may refer to different individuals when sentences containing them are evaluated at different worlds. Yet another system (not treated here) banishes constants from the syntax altogether.[5]

[3]Some of these aspects will not be discussed here.

[4]See Saul A. Kripke, *Naming and Necessity*.

[5]This system is due to Saul A. Kripke, "Semantical Considerations on Modal Logic," *Acta Philosophica Fennica* 16 (1963), 83-94).

If domains of discourse are allowed to vary from world to world, then even if constants are rigid designators, the domain at a world might not contain the object to which it refers. In that case, given the semantics for *PLI*, some sentences will be lacking in truth-values. This forces a change in the semantics, in that "truth-value gaps" must be incorporated into the semantical rules.

If the semantics is going to be changed, we might just as well base the modal system on free logic, which allows for constants that do not refer to any individual in the relevant domain of discourse. This alternative is a natural fit for modal logic, because it provides a way of reasoning about non-existent but possible individuals.

In the next three modules, we will investigate three families of systems of Modal Predicate Logic with identity. The first will be the weakest of such systems, the *Q1R* systems, which are based on free logic. These systems are the most flexible in representing what exists at possible worlds, but they do not have very strong consequences within the logic itself. We will next take a look at the strongest, and conceptually simplest systems, the *Q1* systems, due to Kripke. In these systems, classical predicate logic remains intact, and some quite controversial consequences concerning *de re* and *de dicto* modalities hold in them. Finally, we will look at a family of systems that allows world-relative domains, but based on an underlying system of Predicate Logic which is not "free." The *QPL* systems allow truth-value gaps and are strong enough to forge some interesting connections betwen quantifiers and modal operators in these systems.

2 The Syntax of *MPL*

Before we look at the specific character of the three systems of modal predicate logic, we will lay down the syntax which is common to them all.

The syntax of *MPL* is built upon that of *MSL* and *PL*. Sentence letters, *falsum*, *SL* operators, and modal operators are carried over from *MSL* to *MPL*. Predicate Logic adds constants, variables, predicate letters, the identity predicate symbol, and quantifiers. These expressions were defined in the module on non-modal predicate logic. We can combine the formation rules for Modal Sentential Logic and Predicate Logic to get the following more comprehensive definition of a sentence of Modal Predicate Logic.

1. All sentence letters are sentences of *MPL*.
2. '$\perp$' is a sentence of *MPL*.
3. If $\mathbf{t}$ is a term of *MPL*, then '$\mathrm{E}^1\mathbf{t}$' is a sentence of *MPL*.
4. If $\mathbf{P}^n$ is a predicate of *MPL* and $\mathbf{t_1}, \ldots, \mathbf{t_n}$ are terms of *MPL*, then $\mathbf{P}^n\mathbf{t}_1, \ldots, \mathbf{t}_n$ is a sentence of *MPL*
5. If $\mathbf{t}_1$ and $\mathbf{t}_2$ are terms of *MPL*, then $\mathbf{t}_1 = \mathbf{t}_2$ is a sentence of *MPL*.
6. If α is a sentence of *MPL*, then $\sim\alpha$ is a sentence of *MPL*.
7. If α and β are sentences of *MPL*, then $(\alpha \wedge \beta)$ is a sentence of *MPL*.
8. If α and β are sentences of *MPL*, then $(\alpha \vee \beta)$ is a sentence of *MPL*.
9. If α and β are sentences of *MPL*, then $(\alpha \supset \beta)$ is a sentence of *MPL*.
10. If α and β are sentences of *MPL*, then $(\alpha \equiv \beta)$ is a sentence of *MPL*.
11. If $\alpha(\mathbf{u})$ is a sentence of *MPL* and $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{u})$, then $(\forall\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ is a sentence of *MPL*.

12. If $\alpha(\mathbf{u})$ is a sentence of *MPL* and $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{u})$, then $(\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ is a sentence of *MPL*.
13. If α is a sentence of *MPL*, then $\Box\alpha$ is a sentence of *MPL*.
14. If α is a sentence of *MPL*, then $\Diamond\alpha$ is a sentence of *MPL*.
15. If α and β are sentences of *MPL*, then $\alpha \dashv\!\!\!\!\exists\, \beta$ is a sentence of *MPL*.
16. Nothing else is a sentence of *MPL*.

3 Basic Semantical Principles for *MPL* Semantical Systems

We will here give some semantical definitions and stipulations that are common to all the systems we will study.

3.1 Frames and Interpretations

As with modal sentential logic, the basic semantical structures will be called *frames*. A frame for Modal Predicate Logic results from adding to a Modal Sentential Logic frame $\langle\mathbf{W},\mathbf{R}\rangle$ (at least) a domain of discourse $\mathbf{D}$.[6] So a base *MPL* frame $\mathbf{Fr} = \langle\mathbf{W},\mathbf{R},\mathbf{D}\rangle$. The only restriction on $\mathbf{D}$ is that it be non-empty. (The set of possible worlds $\mathbf{W}$ must also have at least one member.) It may contain infinitely many members. The domain of discourse consists of the "possible objects" to which we can refer using terms of *MPL*. An interpretation based on $\mathbf{Fr}$ just is $\mathbf{Fr}$ with the addition of a valuation function $\mathbf{v_I}$. So a base interpretation $\mathbf{I}$ is (at least) an ordered four-tuple $\langle\mathbf{W},\mathbf{R},\mathbf{D},\mathbf{v}\rangle$.[7]

The valuation function $\mathbf{v_I}$ works just as in the *MSL* semantics for sentence letters, truth-functional sentences, and modal sentences. That is, it is a two-place function from a pair consisting of a sentence and a world to the truth-values $\mathbf{T}$ and $\mathbf{F}$. For syntax proper to *PL*, the first argument of the function is either a constant, a parameter, a predicate, or a *PL* sentence which is not a sentence of *SL*. Thus, the valuation function for *MPL* is an augmented version of the valuation function for **PL**. For example, the valuation function $\mathrm{v_I}$ might return a truth-value $\mathbf{T}$ for the sentence '$\Box Fa$' at a world w: $\mathrm{v_I}(\Box Fa,\mathrm{w}) = \mathbf{T}$.

3.2 Rigid Designation

Since in the semantics for *MPL* we evaluate sentences at worlds, we must deal with the question, raised above, as to the values assigned by the valuation function to constants and parameters. In non-modal Predicate Logic, there is only one domain whose members serve as values of terms for a given interpretation. But now the option is open to allow the assignment of a term at one world to differ from its assignment at another world.

At this point, we will adopt a procedure that will hold for all of the semantical systems to be studied. We shall require that all terms be assigned the same object at all worlds. That is, we will treat terms (both constants and parameters) as rigid designators. For any two worlds, the term has the same value at both worlds.

Rigid Designation
$(\Pi\mathbf{w}_i)(\Pi\mathbf{w}_j)((\mathbf{w}_i \in \mathbf{W} \bigwedge \mathbf{w}_j \in \mathbf{W}) \rightarrow \mathbf{v_I}(\mathbf{t}, \mathbf{w}_i) = \mathbf{v_I}(\mathbf{t}, \mathbf{w}_j))$

[6] In some systems, a frame contains an additional element: a function $\mathbf{q}$ which assigns to each world its own domain.
[7] As noted above, in some systems an additional element is added to each frame.

The assumption of rigid designation greatly simplifies the semantics as well as facilitating completeness results.[8] But the simplification comes at a price. We cannot use constants to stand for definite descriptions, since expressions like "the first human to walk on the moon" would be expected to have different values at different possible worlds. Thus, constants represent only proper names, such as 'Neil Armstrong.'

3.3 Extensions of Predicates

As with non-modal *PL*, the extension of an n-place predicate $\mathbf{P}$ is a set of ordered n-tuples taken from $\mathbf{D}$. Because the valuation function takes a world as an argument, we can allow that the extension of a predicate varies from world to world unlike in *PLI*, where it is fixed. Supposing that $\mathbf{D^w}$ is the domain at a given world $\mathbf{w}$:

Extension of Predicates
$\mathbf{v_I}(\mathbf{P}^n, \mathbf{w}) \subseteq \mathbf{D^w}^n$.

If the extensions of predicates were not allowed to vary from world to world, the resulting semantical system would be of no interest from the standpoint of modality. We would then not be able to distinguish between worlds at which something has a property or stands in a relation and a world at which it does not.

With these basic semantical notions in mind, we now move to the presentation of the systems which employ them.

[8] See Garson, pp. 261-265.