Philosophy 134
Module 2
Non-Modal Sentential Logic

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Contents

The goal of this text is to introduce the reader to some of the main systems of modal logic. It is customary to preface the treatment of modal logics with preliminary accounts of non-modal logics, and this in two ways. First, there is a treatment of *Sentential Logic*, which is followed by a treatment of systems created by adding modal symbolism to the symbolism of Sentential Logic.[1] Second, there is a treatment of *Predicate Logic*, which is followed by systems adding modal symbolism to it.[2] This text will adhere to that custom, though it is quite feasible to introduce modal Predicate Logic from the start.

The logic of sentences or propositions was first investigated by the ancient Stoics, who recognized five rules of inference.[3] It was investigated extensively in the nineteenth century by such logicians as Augustus De Morgan, George Boole, and Ernst Schröder, who explored the relations between logic and algebra.[4] The first presentation in the modern form was by Gottlob Frege[5], and its first in the current notation was by Bertrand Russell and Alfred North Whitehead.[6] The present-day truth-table semantics was developed by Ludwig Wittgenstein[7] and Emil Post.[8] Non-axiomatic, or "natural deduction" systems were introduced by Gerhard Gentzen,[9] and given the form used here by Frederick Fitch.[10]

1 Syntax of Sentential Logic

Sentential Logic (hereinafter called '*SL*,') is a *formal language* which is entirely artificial or conventional. All formal languages have a *syntax*, which determines what symbols belong to the language and how those symbols can be combined to form grammatically correct strings of symbols. Here we will give one formulation of *SL*, with the recognition that there are many variant ways of specifying its symbolism.

1.1 Expressions of *SL*

The formal language *SL* consists of a set of *expressions* (its *vocabulary*) and a set of *rules of formation* which generate a set of *sentences* of *SL*.[11] The expressions of *SL* consist of the following:

Expressions of *SL*

- An infinitely large set of *sentence letters*: $A, B, C, \ldots, Z, A_1, B_1, \ldots, Z_1, A_2, B_2, \ldots$.
- A *sentential constant*: '$\bot$.'
- A set of two *punctuation marks*: '(' and ').'
- A set of five *operators*: '$\sim$,' '$\wedge$,' '$\vee$,' '$\supset$,' and '$\equiv$.'

[1] What is here called "sentential logic" is elsewhere called "sentence logic," "propositional logic," "propositional calculus," "boolean logic," "truth-functional logic," and "the two-valued logic."

[2] What is here called "predicate logic" is elsewhere called "predicate calculus," "quantificational logic," "first-order logic," among other things.

[3] See William and Martha Kneale, *The Development of Logic*, 1962, Chapter III, Section 5.

[4] C. I. Lewis gives a lengthy description of these developments in his *Survey of Symbolic Logic*, 1918, Chapters I-III.

[5] *Begriffsschrift*, 1879

[6] *Principia Mathematica*, 1910

[7] *Tractatus Logico-Philosophicus*, 1921

[8] "Introduction to a General Theory of Elementary Propositions," *The American Journal of Mathematics*, Vll. XLIII (1921), pp. 163-185.

[9] "*Untersuchungen über das logische Schliessen*," *Mathematische Zeitschrift*, Vol. 39 (1934), pp. 176-210.

[10] *Symbolic Logic: An Introduction*, 1952.

[11] The strings of symbols generated by the formation rules are often called "well-formed formulas" or "wffs."

1.2 Rules of Formation for *SL*

In specifying our rules of formation of sentences of *SL*, we use the meta-linguistic variables (or *meta-variables)* 'α', 'β', and 'γ,' with or without primes or positive integer subscripts, to refer to sentences.[12] Thus we will be writing things like, "Suppose $\gamma_1, \ldots, \gamma_n$ are sentences of *SL*." We will refer to sets of sentences of *SL* using upper-case Greek letters, such as 'Γ.' Further, we will distinguish those sentences of *SL* which have no structure as *atomic* sentences from those *compound* sentences which have structure. The former are specified in the first two formation rules, the latter in the next five.

Formation Rules of *SL*

1. All sentence letters are sentences of *SL*.
2. '$\perp$' is a sentence of *SL*.
3. If α is a sentence of *SL*, then $\sim\alpha$ is a sentence of *SL*.[13]
4. If α and β are sentences of *SL*, then $(\alpha \wedge \beta)$ is a sentence of *SL*.
5. If α and β are sentences of *SL*, then $(\alpha \vee \beta)$ is a sentence of *SL*.
6. If α and β are sentences of *SL*, then $(\alpha \supset \beta)$ is a sentence of *SL*.
7. If α and β are sentences of *SL*, then $(\alpha \equiv \beta)$ is a sentence of *SL*.
8. Nothing else is a sentence of *SL*.

A convention we shall employ is that outermost parentheses may be omitted from any sentence which is the end-result of the application of the formation rules. Internal parentheses are always retained.[14]

The definition of a sentence of *SL* is *recursive*. Recursive definitions of sentences of formal languages begin with a listing of basic sentences. For *SL*, the basic sentences are atomic sentences. The second element of a recursive definition of a sentence is a set of rules for generating more complex sentences from simpler ones. So every compound sentence is made up of atomic sentences, to which the rules are applied initially. The results of the intitial application of the rules may then be used as the basis for the formulation of more complex sentences by the application of formation rules to them. The last element of the formation rules is a "closure" clause which states that no string of symbols that is not generated by the rules is a sentence.

The recursive character of the definition of a sentence is of crucial importance for our understanding of the language. In much of what follows, we will be developing some of the *meta-logic* of *SL* and more sophisticated languages. That is, we will be proving that the language and its sentences have certain properties. The meta-logical proofs that we give will generally rely on the recursive character of the definition of a sentence. One proof technique, *mathematical induction* is very powerful, and it derives its power from the structure of the definition of a sentence. We shall soon have occasion to make use of it.

2 Semantics for Sentential Logic

Sentences of *SL* have always been informally intended to abbreviate sentences of natural language, or perhaps to stand in for the propositions which those sentences express. The operators are intended to stand for

[12] These meta-variables will also be used to refer to sentences of the modal languages to follow.

[13] The operators are properly part of the object language, *SL*, but they are also used as names of themselves in meta-linguistic expressions.

[14] Many logic texts employ other conventions which eliminate the display of some or all internal parentheses

grammatical particles of natural language. Thus '$\sim$' is taken to stand for the word 'not' or the phrase 'it is not the case that," and the expressions '$\wedge$' is taken to stand for the word 'and' or the word 'but.' This relation of symbols to natural language is only a first step in developing a system of logic. In such a system, we are primarily interested in how sentences with certain structures are related to other sentences with their structures. More specifially, we are primarily interested in the question of which sentences *follow from* or are *logical consequences of* other sentences.

We could, as do those who produce axiomatic or natural-deduction systems, proceed directly to identify certain sentences and/or patterns of inference as generating correct logical consequences. Another approach, which will be used here, is first to develop in a more formal way the meanings of the sentences of *SL*, by generating a *semantical system* for interpreting them rigorously. This system will give rise to a semantical version of the notion of logical consequence. Then we develop a system based on rules of inference and an axiom system, both of which generate the relations of logical consequence which are in a certain way equivalent.

From a formal standpoint, sentences of *SL* are interpreted as designating exactly one of two *truth-values*, '**T**' and '**F**.' Informally, we think of '**T**' as standing for truth or being true, and '**F**' as standing for falsehood or being false. But in fact, the only thing our semantics requires is that they stand for two distinct objects, perhaps even the letters '**T**' and '**F**' themselves! In computer applications, '**T**' and '**F**' might stand for the state of a transistor being in the "on" position or in the "off" position, respectively.[15] At times in what follows, we will take '**T**' to refer to truth and '**F**' to falsehood.

The sentence letters of *SL* are completely open as far as which truth-value they have. Any sentence letter may be interpreted as having the value **T** or the value **F**. The sentential constant '$\perp$' ("*falsum*") is always interpreted as having the value **F**. The assignment of truth-values for compound sentences is based on an informal understanding of the meanings of the respective operators. The truth-values of compound sentences (i.e., sentences formed using formation rules 3 through 7) are uniquely determined by the truth-values of their component parts. For this reason, the semantics is called *truth-functional*. A *function* may be understood here as a rule which yields no more than one output for every input. In Section 2.3.2 we will prove that the semantical rules for *SL* yield truth-functions.

2.1 Truth-Table Semantics

The most commonly-taught form of semantics for *SL* is based on the truth-table. We will first introduce the interpretations of the sentential operators using truth-table semantics. Later, a more abstract version of the semantics will be given.

A row of a truth-table always has an '**F**' under '*falsum*.' It has at least one of the values **T** or **F**, but not both, under each sentence-letter displayed in it. In addition, it makes has at least one of **T** or **F**, but not both, under compound sentences formed from the displayed sentence-letters. The specific truth-values are based on the following schemata.

2.1.1 *Falsum*

The *falsum* sentential constant has the same value whenever it appears in a truth-table.

$\perp$
F

If *falsum* appears in a sentence of *SL*, it is always assigned **F**, no matter where in the sentence it appears.

[15] Hughes and Cresswell in *A New Introduction to Modal Logic* use the values 1 and 0 rather than **T** and **F**, respectively.

2.1.2 Negation

The *tilde*, '$\sim$' is intended to represent negation.[16] The orthodox view of negation is that it effects a reversal of the truth-value of the negated sentences.[17] So if α has the value **T**, then $\sim\alpha$ has the value **F**. And if α has the value **F**, then $\sim\alpha$ has the value **T**. The semantical behavior of negation is seen graphically in the following truth-table.

α	$\sim\alpha$
T	**F**
F	**T**

2.1.3 Conjunction

The inverted wedge '$\wedge$' is intended to symbolize conjunction.[18] It is generally held that a conjunction is true just in case both its conjuncts are true. Thus, if either conjunct is false, the conjunction is false as well.

α	β	$\alpha \wedge \beta$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

2.1.4 Disjunction

The wedge '$\vee$' is intended to symbolize inclusive disjunction.[19] The standard view about inclusive disjunction is that such a sentence is true just in case at least one of its disjuncts is true.[20] So, if either disjunct is true, the disjunction is true as well.

α	β	$\alpha \vee \beta$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

2.1.5 Material Conditional

The horseshoe '$\supset$' is intended to represent an "if . . . then _ _ _" sentence or conditional.[21] The conditional the horseshoe represents is generally known as the "material" conditional. Sometimes it is said to represent "material implication."[22]

As far as truth-values are concerned, an uncontroversial property of any conditional is that it is false when its antecedent is true and its consequent is false. This gives us one row of a truth-table.

[16] Negation is often indicated by '$\neg$' and sometimes by '–.' Lewis used '–' in his original papers on modal logic, interpreting '$\sim$' as "it is impossible that."

[17] Negation is treated differently in the semantics for intuitionistic and relevant (or "relevance") logics, but this matter is beyond the scope of the current text.

[18] Other conjunction symbols are '&' and '**.**.'

[19] An exclusive disjunction is one in which the disjunction is false if both disjuncts are true.

[20] Some relevant logics recognize an "intensional" form of disjunction which is not truth-functional.

[21] Many texts use an arrow '$\rightarrow$' instead of the horseshoe.

[22] This usage goes back to Russell and Whitehead in *Principia Mathematica*.

α	β	$\alpha \supset \beta$
T	**F**	**F**

In all other cases, the conditional is said to be true. This yields a full truth-table.

α	β	$\alpha \supset \beta$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Most logicians hold that the material conditional is the weakest form of a conditional. That is, the material conditional fulfills the most important (if not the only) necessary condition for any conditional, which is expressed on the second line of the truth-table. It might be held that the material conditional does not fulfill a further necessary condition for a conditional, i.e., that there be some connection between the content of the antecedent and that of the consequent. The truth of the antecedent of a material conditional may not be a "condition" of the truth of the consequent.

Even if one accepts that $\alpha \supset \beta$ expresses a conditional, it seems wrong to say that it expresses any kind of logical implication. This was the objection of C.I. Lewis that led him to invent modern modal logic.[23] In most cases, a true material conditional is true (or formally, has the value **T**) in a way that does not determine whether there is a relation of implication.

Consider two sentence letters, '*A*' and '*B*,' and the following row of a truth-table for a material conditional formed from them:

A	B	$A \supset B$
T	**T**	**T**

On this interpretation, '$A \supset B$' has the value **T**. Should we therefore say that '*A*' implies '*B*'? There is a reason to think that we should not. The fact that '*B*' is assigned **T** has nothing to do with any fact of logic. '*B*' could just as easily be interpreted as having the value **F** given that '*A*' has the value **T**, as is seen from the following partial truth-table:

A	B	$A \supset B$
T	**F**	**F**

On this row of the truth-table, the material conditional has the value **F**, which might lead one to conclude that '*A*' does not imply '*B*.' This is what makes the conditional "material." It is the material facts about the truth-values of its components which determine whether it is true or false. So '*A*' "materially implies" '*B*' in some cases and not in other cases, depending on what factually is the case.

The upshot seems to be that the truth-value of a material conditional on a row of a truth-table is not enough to establish any kind of logical implication, which ought to be independent of any facts that could be either true or false. Perhaps what is needed to express logical implication is that the material conditional be true on all rows of the truth-table. (With this stricter condition, the sentence letter '*A*' does not imply the sentence letter '*B*.') For any two sentences α and β, if $\alpha \supset \beta$ is true on all rows of a truth-table, there is no row on which α has the value **T** and β has the value **F**.

It is worth exploring this suggestion a little further. It is easy to establish by truth-tables that there is no row on the truth-table in which '*A*' is true and '$B \supset A$' is false. So we might wish to say that '*A*' strictly

[23]See E. M. Curley, "The Development of Lewis' Theory of Strict Implication," *Notre Dame Journal of Formal Logic* Vol. XVI, No. 4, pp. 517-530.

implies '$B \supset A$.' Some people understand this to mean that a true sentence is implied by every sentence. That is, if 'A' is true, 'A' is implied by 'B' because '$B \supset A$' is true. If '$B \supset A$' is taken to express implication, then it would be the case that a true sentence 'A' is implied by any sentence. But in fact, the only kind of "implication" we have with '$A \supset B$' is "material implication," as we have seen. Since 'A' and 'B' are sentence letters, there is a row on a truth-table on which 'A' is true and 'B' is false. It is a trivial fact of the semantics that a true sentence is "materially implied" by any sentence, and this is all that is contained in the claim that a true sentence is implied by every sentence.

A further consideration in this connection is the fact that '$B \supset A$' is equivalent in the semantics to '$\sim B \vee A$.' As Lewis pointed out, this loose connection by inclusive disjunction seems hardly strong enough to warrant the title "implication."

2.1.6 Material Biconditional

The material biconditional, expressed by the triple-bar '≡,' has the same truth-table as the conjunction of two material conditionals.[24] Such a biconditional is true just in case its components have the same truth-value.

α	β	$\alpha \equiv \beta$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Whether the triple-bar should be understood as a genuine biconditional or as expressing co-implication hinges on whether the material conditional should be understood as a genuine conditional or as expressing implication, so it will not be discussed further here.

2.2 Formal Semantics for *SL*

The truth-table method for determining the truth-values of sentences of *SL* has been presented in an informal way. In what follows, the semantics for *SL* will be presented through the use of explicit definitions and rules. This way of approaching the semantics of *SL* (and richer languages) will be referred to as *formal semantics*.

The core notion in the formal semantics for Sentential Logic is that of an *interpretation*. An *SL*-interpretation is defined recursively. We describe a set of *semantical rules* which govern any interpretation. The base component is a collective assignment of truth-values (or *truth-value assignment*, TVA) to all the sentence-letters of *SL*. The TVA made by an interpretation gives to each sentence-letter either the value **T** or the value **F**, but not both, as is done for truth-tables.[25] The other component is a set of semantical rules which determine the truth-values of *falsum* and the compound sentences of *SL*. These rules can be seen as formal ways of specifying how to determine truth-values in the columns under compound sentences and *falsum* in truth-tables.

A feature of the semantical rules as stated here is that they are biconditionals. In reasoning through truth-tables, we are often in a position where we wish to determine the truth-values of simpler components of a compound sentence. Thus we say that if $\alpha \wedge \beta$ has the value **T**, then β has the value **T**. This "reverse" reasoning will prove to be crucial to proving some important meta-logical properties of *SI*.

[24] Sometimes the material biconditional is symbolized as '↔.'

[25] We shall call such an assignment a *complete* TVA, which covers infinitely many sentence letters. In providing specfic truth-tables, we work with *partial* truth-value assignments, which leave out the values of expressions of *SL* not found in the sentences under evaluation.

2.2.1 Semantical Rules for *SL*

The meta-variable '**I**' (with or without primes or positive integer subscripts) will be used to refer to an unspecified or arbitrary interpretation. **I** itself is an ordered set whose only member is a function which maps sentence-letters to truth-values. We shall call such a function a *valuation-function* and represent it with the meta-variable '**v**' (with or without primes or positive integer subscripts). Thus $\mathbf{I} = \langle \mathbf{v} \rangle$.[26] Since different valuation-functions will be associated with different interpretations, we will label the valuation-function **v** for interpretation **I** as '$\mathbf{v_I}$.' To state that an unspecified interpretation **I** assigns the value **T** to an unspecified sentence α we write:

$\mathbf{v_I}(\alpha)=\mathbf{T}$.

For a specific sentence of *SL*, $A \wedge B$, which has the value **T** based on **I**, we would write:

$\mathbf{v_I}(A \wedge B)=\mathbf{T}$.

We may now specify in formal notation the semantical rules governing valuation-functions in Sentential Logic.

SR-TVA If α is a sentence-letter, then either $\mathbf{v_I}(\alpha)=\mathbf{T}$ or $\mathbf{v_I}(\alpha)=\mathbf{F}$; it is not the case that $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\alpha)=\mathbf{F}$.

SR-⊥ For all I,$\mathbf{v_I}(\bot)=\mathbf{F}$ and $\mathbf{v_I}(\bot)\neq\mathbf{T}$.

SR-∼ $\mathbf{v_I}(\sim\alpha)=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{F}$; $\mathbf{v_I}(\sim\alpha)=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{T}$.

SR-∧ $\mathbf{v_I}(\alpha \wedge \beta)=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\beta)=\mathbf{T}$; $\mathbf{v_I}(\alpha \wedge \beta)=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{F}$ or $\mathbf{v_I}(\beta)=\mathbf{F}$.

SR-∨ $\mathbf{v_I}(\alpha \vee \beta)=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{T}$, or $\mathbf{v_I}(\beta)=\mathbf{T}$; $\mathbf{v_I}(\alpha \vee \beta)=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{F}$ and $\mathbf{v_I}(\beta)=\mathbf{F}$.

SR-⊃ $\mathbf{v_I}(\alpha \supset \beta)=\mathbf{T}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{F}$ or $\mathbf{v_I}(\beta)=\mathbf{T}$; $\mathbf{v_I}(\alpha \supset \beta)=\mathbf{F}$ if and only if $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\beta)=\mathbf{F}$.

SR-≡ $\mathbf{v_I}(\alpha \equiv \beta)=\mathbf{T}$ if and only if either $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\beta)=\mathbf{T}$, or $\mathbf{v_I}(\alpha)=\mathbf{F}$ and $\mathbf{v_I}(\beta)=\mathbf{F}$; $\mathbf{v_I}(\alpha \equiv \beta)=\mathbf{F}$ if and only if either $\mathbf{v_I}(\alpha)=\mathbf{T}$ and $\mathbf{v_I}(\beta)=\mathbf{F}$, or $\mathbf{v_I}(\alpha)=\mathbf{F}$ and $\mathbf{v_I}(\beta)=\mathbf{T}$.

The semantical rules taken together make up a *semantical system* for *SL*. We will call the present system "*SI*" (for "sentential, interpretation"). When referring to a specific interpretation, we shall use the non-boldface letter 'I' (with or without primes or positive integer subscripts). Similarly, we will use the non-boldface letter 'v' to refer to a specific valuation function. For example, to say that a valuation function v in an interpretation I makes '*A*' true, we write: $\mathrm{v_I}(A)=\mathbf{T}$.

2.3 Semantical Properties and Relations

Having given formal definitions for the truth or falsehood of sentences of *SL*, we may now turn to some of the important semantical properties and relations of *SL* sentences. These properties and relations follow directly from the semantical rules which make up the semantical system *SI*, which closely parallel the formation rules for sentences of the language. We will engage in meta-logical reasoning in proving that the properties and relations hold. The results of our reasoning will be *meta-theorems*. We will also be using this kind of reasoning for other purposes.

[26] The reason for representing **I** as an ordered one-tuple will become apparent when the semantics is extended to cover modal sentences.

In most logic texts, meta-logical reasoning is done informally. We will do some informal semantical reasoning in this text, but in many cases, our reasoning will be of a more formal sort. Specifically, we will employ *meta-logical derivations* to reason to our conclusions. These derivations will be in predicate logic, and they will appeal to the semantical rules as well as to the rules of inference of predicate logic. In our meta-logical reasoning, we will be using meta-logical symbols which correspond to English expressions.

Meta-Logical Symbols

Not	$\neg$
And	$\bigwedge$
Or	$\bigvee$
Only if	$\rightarrow$
If and only if	$\leftrightarrow$
For all	Π
There is	Σ

We will also use generally accepted rules of inference and annotations for them. The style of the proofs will for the most part parallel the way truth-tables might be used to argue for the same results.

2.3.1 Bivalence

The first meta-logical property we will investigate is *bivalence*: every sentence of *SL* has either the value **T** or **F**.

BV: $\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$

The semantical rules stipulate that all the atomic sentences have either the value **T** or the value **F**. So it remains to be shown that all compound sentences have one of these two values. The argument is that if all of the components of a compound sentence have a truth-value, the compound sentence itself has a truth value. This can be seen by considering all the operators. If α has the value **T**, then $\sim\alpha$ has the value **F**, and if α has the value **F**, then $\sim\alpha$ has the value **T**. The cases of the two-place operators are the same. The semantical rules guarantee that if the sentences joined by the operators have truth-values, then so does the compound sentence. So, since all the atomic sentences have truth-values, any compound sentence generated from them has a truth-value, and any compound sentences generated from the resulting compound sentences have truth-values, etc. This argument is an informal application of mathematical induction.

We will now undertake part of a formal proof by mathematical induction. A proof by mathematical induction has three parts. The first is the *basis step*, where the property is shown to hold for some set of objects, which in the proofs to follow will be atomic sentences. Next, an *inductive hypothesis* is made. It is assumed that the property in question holds for all objects with less than a specified level of complexity *n*. In the proofs to follow, *n* will be the number of operators in a sentence. Finally, in an *induction step* it is shown that the property holds for objects with complexity *n*. With this, the proof is concluded, as we have shown that the property holds for objects of any complexity. In our meta-logical proofs we will routinely omit noting that the reasoning applies to all interpretations because the choice of **I** in the proof is arbitrary.

Proof of Bivalence by mathematical induction on the number of connective *n* of α.

Basis Step. $n = 0$. **Case 1**. α is a sentence letter. By **SR-TVA**, every sentence letter has at least one truth-value, so $\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$. **Case 2**. α is $\perp$. Since $\mathbf{v_I}(\perp) = \mathbf{F}$, it follows that $\mathbf{v_I}(\perp) = \mathbf{T} \bigvee \mathbf{v_I}(\perp) = \mathbf{F}$.

Inductive Hypothesis. Suppose that for all sentences α with fewer than *n* operators, $\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$.

Induction Step. Let α contain n operators. The goal is to show that $\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$. We must consider five cases, which reflect what α might be.

Case 1. α is $\sim\beta$.

1	$\mathbf{v_I}(\beta) = \mathbf{T} \bigvee \mathbf{v_I}(\beta) = \mathbf{F}$	Inductive hypothesis
2	$\mid\ \underline{\mathbf{v_I}(\beta) = \mathbf{T}}$	Assumption
3	$\mid\ \mathbf{v_I}(\sim\beta) = \mathbf{F}$	1 **SR-**$\sim$
4	$\mid\ \mathbf{v_I}(\sim\beta) = \mathbf{T} \bigvee \mathbf{v_I}(\sim\beta) = \mathbf{F}$	3 $\bigvee$ I
5	$\mid\ \underline{\mathbf{v_I}(\beta) = \mathbf{F}}$	Assumption
6	$\mid\ \mathbf{v_I}(\sim\beta) = \mathbf{T}$	5 **SR-**$\sim$
7	$\mid\ \mathbf{v_I}(\sim\beta) = \mathbf{T} \bigvee \mathbf{v_I}(\sim\beta) = \mathbf{F}$	6 $\bigvee$ I
8	$\mathbf{v_I}(\sim\beta) = \mathbf{T} \bigvee \mathbf{v_I}(\sim\beta) = \mathbf{F}$	1 2-4 5-7 $\bigvee$ E

Case 2. α is $\beta \wedge \gamma$.

1	$\mathbf{v_I}(\beta) = \mathbf{T} \bigvee \mathbf{v_I}(\beta) = \mathbf{F}$	Inductive hypothesis
2	$\mathbf{v_I}(\gamma) = \mathbf{T} \bigvee \mathbf{v_I}(\gamma) = \mathbf{F}$	Inductive hypothesis
3	$\mid\ \underline{\mathbf{v_I}(\beta) = \mathbf{T}}$	Assumption
4	$\mid\ \mid\ \underline{\mathbf{v_I}(\gamma) = \mathbf{T}}$	Assumption
5	$\mid\ \mid\ \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T}$	3 4 **SR-**$\wedge$
6	$\mid\ \mid\ \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigvee \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F}$	5 $\bigvee$ I
7	$\mid\ \mid\ \underline{\mathbf{v_I}(\gamma) = \mathbf{F}}$	Assumption
8	$\mid\ \mid\ \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F}$	7 **SR-**$\wedge$
9	$\mid\ \mid\ \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigvee \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F}$	8 $\bigvee$ I
10	$\mid\ \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigvee \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F}$	2 4-6 7-9 $\bigvee$ E
11	$\mid\ \underline{\mathbf{v_I}(\beta) = \mathbf{F}}$	Assumption
12	$\mid\ \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F}$	11 **SR-**$\wedge$
13	$\mid\ \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigvee \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F}$	12 $\bigvee$ I
14	$\mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigvee \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F}$	1 3-10 11-13 $\bigvee$ E

Case 3. α is $\beta \vee \gamma$.

Case 4. α is $\beta \supset \gamma$.

Case 5. α is $\beta \equiv \gamma$.

The last three cases may be completed as an exercise.

So we have shown that given the assumption that sentences with fewer than n operators have at least one truth-value, any sentence with n operators, no matter how it is formed, has at least one truth-value. And since all sentences with 0 operators have at least one truth-value, we can conclude that no matter how complex a sentence, it must have at least one truth-value.

2.3.2 Truth-Functionality

It was noted earlier that the operators as interpreted in *SI* are truth-functional. If one begins with a sentence or two sentences having a single truth-value, the sentence resulting from the application of the formation rule for a given operator will have only a single truth-value. In semantical system *SI*, for any sentence α, and any interpretation **I**, it is not the case that α has both the value **T** and the value **F**.

TF: $\neg(\mathbf{v_I}(\alpha) = \mathbf{T} \bigwedge \mathbf{v_I}(\alpha) = \mathbf{F})$.

We will say that because no sentence has more than one truth-value, *SI* is *truth-functional*. A partial proof **TF** will be given. The last three cases can be completed as an exercise.

Proof of Truth-Functionality by mathematical induction on the number of connectives n of α.

Basis Step. $n = 0$. **Case 1**. α is a sentence letter. By **SR-TVA**, no sentence letter has more than one truth-value, so $\neg(\mathbf{v_I}(\alpha) = \mathbf{T} \bigwedge \mathbf{v_I}(\alpha) = \mathbf{F})$. **Case 2**. α is $\bot$. Since for any **I**, $\mathbf{v_I}(\bot) = \mathbf{F}$ and never **T**, it follows that $\neg(\mathbf{v_I}(\bot) = \mathbf{T} \bigwedge \mathbf{v_I}(\bot) = \mathbf{F})$.

Inductive Hypothesis. Suppose that for all sentences α with fewer than n operators, $\neg(\mathbf{v_I}(\alpha) = \mathbf{T} \bigwedge \mathbf{v_I}(\alpha) = \mathbf{F})$.

Induction Step. Let α contain n operators. We may distinguish five cases.

Case 1. α is $\sim\beta$. (In the proofs of this and the next case, we will use a derived meta-logical rule which we will call "Negated Conjunction." From $\sim\alpha$ we may infer $\sim(\alpha \wedge \beta)$.)

1	$\neg(\mathbf{v_I}(\beta) = \mathbf{T} \bigwedge \mathbf{v_I}(\beta) = \mathbf{F})$	Inductive hypothesis
2	$\neg\mathbf{v_I}(\beta) = \mathbf{T} \bigvee \neg\mathbf{v_I}(\beta) = \mathbf{F}$	1 De Morgan's
3	$\mathbf{v_I}(\beta) = \mathbf{T} \bigvee \mathbf{v_I}(\beta) = \mathbf{F}$	**BV**
4	$\quad\mid\ \mathbf{v_I}(\beta) = \mathbf{T}$	Assumption
5	$\quad\mid\ \neg\neg\mathbf{v_I}(\beta) = \mathbf{T}$	4 Double Negation
6	$\quad\mid\ \neg\mathbf{v_I}(\beta) = \mathbf{F}$	2 5 Disjunctive Syllogism
7	$\quad\mid\ \neg\mathbf{v_I}(\sim\beta) = \mathbf{T}$	6 **SR-**~
8	$\quad\mid\ \neg(\mathbf{v_I}(\sim\beta) = \mathbf{T} \bigwedge \mathbf{v_I}(\sim\beta) = \mathbf{F})$	7 Negated Conjunction
9	$\quad\mid\ \mathbf{v_I}(\beta) = \mathbf{F}$	Assumption
10	$\quad\mid\ \neg\neg\mathbf{v_I}(\beta) = \mathbf{F}$	9 Double Negation
11	$\quad\mid\ \neg\mathbf{v_I}(\beta) = \mathbf{T}$	2 10 Disjunctive Syllogism
12	$\quad\mid\ \neg\mathbf{v_I}(\sim\beta) = \mathbf{F}$	11 **SR-**~
13	$\quad\mid\ \neg(\mathbf{v_I}(\sim\beta) = \mathbf{T} \bigwedge \neg\mathbf{v_I}(\sim\beta) = \mathbf{F})$	12 Negated Conjunction
14	$\neg(\mathbf{v_I}(\sim\beta) = \mathbf{T} \bigwedge \neg\mathbf{v_I}(\sim\beta) = \mathbf{F})$	3 4-8 9-13 $\bigvee$ E

Exercise. Prove the result by assuming the negation of the conclusion.

Case 2. α is $\beta \wedge \gamma$.

1	$\neg(\mathbf{v_I}(\beta) = \mathbf{T} \bigwedge \mathbf{v_I}(\beta) = \mathbf{F})$	Inductive hypothesis
2	$\neg(\mathbf{v_I}(\gamma) = \mathbf{T} \bigwedge \mathbf{v_I}(\gamma) = \mathbf{F})$	Inductive hypothesis
3	$\mathbf{v_I}(\beta) = \mathbf{T} \bigvee \mathbf{v_I}(\beta) = \mathbf{F}$	**BIV**
4	$\mathbf{v_I}(\gamma) = \mathbf{T} \bigvee \mathbf{v_I}(\gamma) = \mathbf{F}$	**BIV**
5	$\neg\mathbf{v_I}(\beta) = \mathbf{T} \bigvee \neg\mathbf{v_I}(\beta) = \mathbf{F}$	1 De Morgan's
6	$\neg\mathbf{v_I}(\gamma) = \mathbf{T} \bigvee \neg\mathbf{v_I}(\gamma) = \mathbf{F}$	2 De Morgan's
7	$\mathbf{v_I}(\beta) = \mathbf{T}$	Assumption
8	$\mathbf{v_I}(\gamma) = \mathbf{T}$	Assumption
9	$\neg\neg\mathbf{v_I}(\gamma) = \mathbf{T}$	8 Double Negation
10	$\neg\mathbf{v_I}(\gamma) = \mathbf{F}$	6 9 Disjunctive Syllogism
11	$\neg\neg\mathbf{v_I}(\beta) = \mathbf{T}$	7 Double Negation
12	$\neg\mathbf{v_I}(\beta) = \mathbf{F}$	5 11 Disjunctive Syllogism
13	$\neg\mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F}$	10 12 **SR-**$\wedge$
14	$\neg(\mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigwedge \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F})$	13 Negated Conjunction
15	$\mathbf{v_I}(\gamma) = \mathbf{F}$	Assumption
16	$\neg\neg\mathbf{v_I}(\gamma) = \mathbf{F}$	15 Double Negation
17	$\neg\mathbf{v_I}(\gamma) = \mathbf{T}$	6 16 Disjunctive Syllogism
18	$\neg\mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T}$	17 **SR-**$\wedge$
19	$\neg(\mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigwedge \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F})$	18 Negated Conjunction
20	$\neg(\mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigwedge \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F})$	4 8-14 15-19 $\bigvee$ E
21	$\mathbf{v_I}(\beta) = \mathbf{F}$	Assumption
22	$\neg\neg\mathbf{v_I}(\beta) = \mathbf{F}$	21 Double Negation
23	$\neg\mathbf{v_I}(\beta) = \mathbf{T}$	5 22 Disjunctive Syllogism
24	$\neg\mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T}$	23 **SR-**$\wedge$
25	$\neg(\mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigwedge \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F})$	24 Negated Conjunction
26	$\neg(\mathbf{v_I}(\beta \wedge \gamma) = \mathbf{T} \bigwedge \mathbf{v_I}(\beta \wedge \gamma) = \mathbf{F})$	3 7-20 21-25 $\bigvee$ E

Case 3. α is $\beta \vee \gamma$.

Case 4. α is $\beta \supset \gamma$.

Case 5. α is $\beta \equiv \gamma$.[27]

[27] The reader might notice that the use of semantical rules in these derivations requires their statement as biconditionals. For example, in the proof of Case 3, we appeal in step 13 to the contrapositive of the claim that if $\beta \wedge \gamma$ is false, then β is false.

So we have shown that given the assumption that sentences with fewer than n operators have no more than one truth-value, any sentence with n operators, no matter how it is formed, has no more than one truth-value. And since all sentences with 0 operators have no more than one truth-value, we can conclude that no matter how complex a sentence, it must have no more than one truth-value. Combining Truth-Functionality with Bivalence, we get the result that all sentences of *SL* have exactly one truth-value when interpreted through the semantical machinery of *SI*.

It was noted in Section 2 that the operators as interpreted in *SI* are truth-functional. The proof of Truth-Functionality establishes this point. If one begins with a sentence or two sentences having a single truth-value, the sentence resulting from their combination by way of a connective will have only a single truth-value.

2.3.3 Semantical Entailment

The relation of *semantical entailment* holds between a set of sentences and a single sentence.[28] The set of sentences $\{\gamma_1, \cdots, \gamma_n\}$ semantically entails a sentence α if and only if on all interpretations **I** on which $\mathbf{v_I}(\gamma_1)$=**T**, and . . . and $\mathbf{v_I}(\gamma_n)$=**T**, $\mathbf{v_I}(\alpha)$ is also **T**. We symbolize semantical entailment using the subscripted double-turnstyle '$\vDash_{SI}$' between $\{\gamma_1, \cdots, \gamma_n\}$ and α: $\{\gamma_1 \ldots \gamma_n\} \vDash_{SI} \alpha$.[29]

Semantical Entailment. $\{\gamma_1 \ldots \gamma_n\} \vDash_{SI} \alpha$ if and only if for all interpretations **I**, if $\mathbf{v_I}(\gamma_1)$=**T**, and . . . and $\mathbf{v_I}(\gamma_n)$=**T**, then $\mathbf{v_I}(\alpha)$ =**T**

One typically proves that a semantical entailment holds by assuming for an arbitrary interpretation **I** that the value given by $\mathbf{v_I}$ to each γ_i is **T** and then using the semantical rules or their consequences to show that the value of α is **T** as well. Suppose, for example, that $\mathbf{v_I}(\alpha) = \mathbf{T}$ and $\mathbf{v_I}(\alpha \supset \beta) = \mathbf{T}$. By Truth-Functionality, it is not the case that $\mathbf{v_I}(\alpha) = \mathbf{T}$ and $\mathbf{v_I}(\alpha) = \mathbf{F}$. Therefore, it is not the case that $\mathbf{v_I}(\alpha) = \mathbf{F}$. By **SR-⊃**, $\mathbf{v_I}(\alpha) = \mathbf{F}$ or $\mathbf{v_I}(\alpha) = \mathbf{T}$. Then by Disjunctive Syllogism, $\mathbf{v_I}(\alpha) = \mathbf{T}$.

Semantical proof that: $\alpha, \alpha \supset \beta \vDash_{SI} \beta$

1	$\mathbf{v_I}(\alpha) = \mathbf{T}$	Assumption
2	$\mathbf{v_I}(\alpha \supset \beta) = \mathbf{T}$	Assumption
3	$\neg(\mathbf{v_I}(\alpha) = \mathbf{T} \bigwedge \mathbf{v_I}(\alpha) = \mathbf{F})$	**TF**
4	$\neg\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \neg\mathbf{v_I}(\alpha) = \mathbf{F}$	3 De Morgan's
5	$\neg\neg\mathbf{v_I}(\alpha) = \mathbf{T}$	1 Double Negation
6	$\neg\mathbf{v_I}(\alpha) = \mathbf{F}$	4 5 Disjunctive Syllogism
7	$\mathbf{v_I}(\alpha \supset \beta) = \mathbf{T} \rightarrow (\mathbf{v_I}(\alpha) = \mathbf{F} \bigvee \mathbf{v_I}(\beta) = \mathbf{T})$	**SR-⊃**
8	$\mathbf{v_I}(\alpha) = \mathbf{F} \bigvee \mathbf{v_I}(\beta) = \mathbf{T}$	2 7 Modus Ponens
9	$\mathbf{v_I}(\beta) = \mathbf{T}$	6 8 Disjunctive Syllogism

[28]This technical notion must be distinguished from more vague, intuitive notions of entailment, which may be stronger or weaker than semantical entailment.

[29]The subscript indicates the semantical system *SI*. The subscript will be dispensed with in contexts where it is clear which semantical system is intended.

2.3.4 Semantical Equivalence

Two sentences α and β are *semantically equivalent* just in case they have the same truth-value on all interpretations.[30]

Semantical Equivalence. α is semantically equivalent to β if and only if for all interpretations $\mathbf{I}$, $\mathbf{v_I}(\alpha) = \mathbf{v_I}(\beta)$.

The relation of semantical equivalence holds for sentences α and β just in case $\{\alpha\} \vDash_{SI} \beta$ and $\{\beta\} \vDash_{SI} \alpha$. Suppose that $\{\alpha\} \vDash_{SI} \beta$ and $\{\beta\} \vDash_{SI} \alpha$. Then any interpretation $\mathbf{I}$ that assigns α the value $\mathbf{T}$ assigns β the value $\mathbf{T}$. Further, any interpretation $\mathbf{I}$ that assigns β the value $\mathbf{T}$ assigns α the value $\mathbf{T}$. So on any interpretation $\mathbf{I}$, $\mathbf{v_I}(\alpha) = \mathbf{T}$ if and only if $\mathbf{v_I}(\beta) = \mathbf{T}$. By Truth-Functionality, it follows that on any interpretation $\mathbf{I}$, $\mathbf{v_I}(\alpha) = \mathbf{F}$ if and only if $\mathbf{v_I}(\beta) = \mathbf{F}$. So given that $\{\alpha\} \vDash_{SI} \beta$ and $\{\beta\} \vDash_{SI} \alpha$, on any interpretation, α and β have the same truth-value, in which case they are semantically equivalent.

Exercise. Prove the converse.

2.3.5 Validity

The limiting case of semantical entailment, $\Gamma \vDash_{SI} \alpha$, is that in which Γ contains no sentences at all. The set of sentences is then $\emptyset$, the empty set. If $\emptyset \vDash \alpha$, then on all interpretations $\mathbf{I}$, α is true. In that case, we say that α is *valid in SI* (or *SI-valid*).[31] If α is *SI*-valid, we write '$\vDash_{SI} \alpha$.'

Validity. $\vDash_{SI} \alpha$ if and only if on all interpretations I, $\mathbf{v_I}(\alpha) = \mathbf{T}$.

For example, '$A \vee {\sim}A$' is valid in *SI*. Consider an arbitrary interpretation $\mathbf{I}$. By Bivalence, $\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$. Now suppose $\mathbf{v_I}(\alpha) = \mathbf{T}$. It follows from **SR-**$\vee$ that $\mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$. And suppose that $\mathbf{v_I}(\alpha) = \mathbf{F}$. In that case, by **SR-**$\sim$ it follows that $\mathbf{v_I}({\sim}\alpha) = \mathbf{T}$. Therefore, by **SR-**$\vee$, $\mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$. So in either case we have the result that $\alpha \vee {\sim}\alpha$ has the value $\mathbf{T}$ in $\mathbf{I}$, and since the choice of I is arbitrary, the result holds for all interpretations.

Semantical proof that: $\vDash_{SI} \alpha \vee {\sim}\alpha$

1	$\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$	**BV**
2	$\quad \mathbf{v_I}(\alpha) = \mathbf{T}$	Assumption
3	$\quad \mathbf{v_I}(\alpha) = \mathbf{T} \rightarrow \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	**SR-**$\vee$
4	$\quad \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	2 3 $\rightarrow$ E
5	$\quad \mathbf{v_I}(\alpha) = \mathbf{F}$	Assumption
6	$\quad \mathbf{v_I}(\alpha) = \mathbf{F} \rightarrow \mathbf{v_I}({\sim}\alpha) = \mathbf{T}$	**SR-**$\sim$
7	$\quad \mathbf{v_I}({\sim}\alpha) = \mathbf{T}$	5 6 $\rightarrow$ E
8	$\quad \mathbf{v_I}({\sim}\alpha) = \mathbf{T} \rightarrow \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	**SR-**$\vee$
9	$\quad \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	7 8 $\rightarrow$ E
10	$\mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	1 2-4 5-9 $\bigvee$ E

[30] There is no generally accepted meta-logical symbol to denote semantic equivalence.

[31] Such a sentence of Sentential Logic is often referred to as a "tautology." It is also called a "logical truth," though that notion seems to be more generic than that of a tautology, as sentences of predicate logic and modal logic are also called logical truths.

The relation of semantical entailment and the property of validity, if they hold, can be established using truth-tables. In effect, the semantical reasoning we have been using can be mapped onto the construction of a truth-table.

α	$\sim\alpha$	$\alpha \vee \sim\alpha$
T		**T**
F	**T**	**T**

Steps 1 and 2 of the meta-logical deduction correspond to the placing of '**T**' and '**F**' respectively in the two rows under 'α.' Step 4 corresponds to the placing of the '**T**' under '$\alpha \vee \sim\alpha$' in the first row. Step 7 corresponds to the placing of '**T**' under '$\sim\alpha$' in the second row. Step 9 corresponds to the placing of '**T**' under '$\alpha \vee \sim\alpha$' in the second row. (There is no need to calculate a value for $\sim\alpha$ in the first row.)

2.3.6 Semantical Consistency

We may define a notion of *semantical consistency*, which applies to a set of sentences Γ. Specifically, Γ is semantically consistent (*s-consistent*) if and only if there is an interpretation **I** such that all the sentences of Γ have the value **T** on *I*. A sentence which is not s-consistent is *s-inconsistent*. A set Γ is s-inconsistent if and only if there is no interpretation on which all the sentences of Γ have the value **T**, which by Bivalence is equivalent to saying that on all interpretations, at least one member of Γ has the value **F**.

Semantical Consistency. Γ is semantically consistent if and only if there is an interpretation *I* such that for all γ_i in Γ, $\mathbf{v_I}(\gamma_1)$=**T**, and . . . and $\mathbf{v_I}(\gamma_n)$=**T**; Γ is semantically inconsistent if and only if there is no interpretation **I** such that for all γ_i in Γ, $\mathbf{v_I}(\gamma_1)$=**T**, and . . . and $\mathbf{v_I}(\gamma_n)$=**T**.

For example, the set of sentences $\{A, A \supset B, B\}$ is s-consistent, while the set $\{A, A \supset B, \sim B\}$ is s-inconsistent.

There is a close relation between s-inconsistency and semantical entailment. Let $\Gamma \cup \sim\alpha$ be the *union* of the set of sentences Γ and the sentence $\sim\alpha$.[32] The following meta-theorem may now be proved.

$\Gamma \vDash_{SI} \alpha$ if and only if $\Gamma \cup \sim\alpha$ is s-inconsistent.

$\Gamma \vDash_{SI} \alpha$ if and only if for any interpretation **I**, and any γ_i in Γ, if $\mathbf{v_I}(\gamma_i) = \mathbf{T}$ then $\mathbf{v_I}(\alpha) = \mathbf{T}$. By Truth-Functionality, this holds if and only if, if $\mathbf{v_I}(\gamma_i) = \mathbf{T}$ then $\neg\mathbf{v_I}(\alpha) = \mathbf{F}$. By **SR-**$\sim$, this holds if and only if, if $\mathbf{v_I}(\gamma_i) = \mathbf{T}$, then $\neg\mathbf{v_I}(\sim\alpha) = \mathbf{T}$. Finally, this result holds if and only if there is no interpretation **I** such that all the members γ_i of Γ have the value **T** and $\sim\alpha$ has the value **T**, i.e. $\Gamma \cup \sim\alpha$ is s-inconsistent.

3 Natural Deduction in Sentential Logic

As was noted in Module 1, Frederick Fitch devised a set of "natural deduction" rules that show which sentences "follow from" which sets of sentences. These rules are purely syntactical, in the sense that they do not rely (at least explicitly) on any interpretation of the sentences to which they are applied. The end-result of the use of the rules is a *derivation* of a conclusion from zero or more premises. We will begin by using a fairly restrictive set Introduction and Elimination rules which produce a derivation system we call '*SD*' (for "sentential derivations"), following Bergmann, Moore, and Nelson in *The Logic Book*. The system *SD* in *The Logic Book* does not contain rules for *falsum*, which is not an expression in their language of sentential logic. For each operator other than *falsum*, there will be one rule for its "introduction" and one for its "elimination."[33] We will allow the use of two additional, derived, rules for *falsum*.

[32] The union of two sets is the set consisting of all members of both sets.

[33] Gerhard Gentzen, who originally formulated introduction/elimination rules, found philosophical significance in the symmetry between introduction and elimination rules.

The first step in any Fitch-style proof is setting down one or more *assumptions*. A vertical line known as the *scope line* is drawn next to the assumption(s) and a horizontal line is drawn below the final assumption.

γ_1	Assumption
$\vdots$	
γ_n	Assumption

Certain rules allow the assumption to be *discharged*. When an assumption is discharged, the scope line is ended. An assumption that has not, at a given point in the derivation, been discharged is called an *undischarged assumption*. A sentence α to the right of a scope line is said to be *in the scope* of the assumption (which may be α itself) made at that scope line. A sentence in the scope of an assumption and not in the scope of any further assumption to its right is said to be in its *immediate* scope.

3.1 Reiteration

The rule of *Reiteration* (R) allows any sentence already arrived at in a derivation to be repeated at any step below it in its immediate scope or in the scope of an assumption made in its immediate scope. We shall require that the rest of the rules be used entirely within a single scope line. Thus Reiteration may have to be applied once or more in order to allow a rule to be used. Although this restriction is of no real value in sentential logic, it is very helpful for derivations in modal logic, which make use of special modal scope lines.

Reiteration

α	Already Derived
$\vdots$	
α	R
| β	Assumption
| $\vdots$	
| α	R

This schematic representation of the rule incorporates both kinds of permissible reiterations. One need not make use of both kinds in all derivations.

3.2 Negation Rules

The rules for introducing and eliminating negation operators (~ I and ~ E, respectively) are quite similar to each other. In each case, an assumption is made and a pair of sentences β and $\sim\beta$ is derived (in either order) within the immediate scope of that assumption. At that point, the scope line ends and a negation sign is either added to or removed from the assumption (depending on whether the introduction or elimination rule is being used).

Negation Introduction

$\vert\ \alpha$	Assumption
$\vert\ \vdots$	
$\vert\ \beta$	
$\vert\ \vdots$	
$\vert\ {\sim}\beta$	
${\sim}\alpha$	$\sim$ I

Negation Elimination

$\vert\ {\sim}\alpha$	Assumption
$\vert\ \vdots$	
$\vert\ \beta$	
$\vert\ \vdots$	
$\vert\ {\sim}\beta$	
α	$\sim$ E

3.3 Conjunction Rules

The conjunction rules are perhaps the most straightforward of all. A conjunction can be introduced by conjoining two sentences (in either order) in the same scope line (by $\wedge$ I).

Conjunction Introduction

$\vert\ \alpha$	Already Derived
$\vert\ \vdots$	
$\vert\ \beta$	Already Derived
$\vert\ \vdots$	
$\vert\ \alpha \wedge \beta$	$\wedge$ I
$\vert\ \vdots$	
$\vert\ \beta \wedge \alpha$	$\wedge$ I

A conjunction can be eliminated (using $\wedge$ E) by writing down one of its conjuncts in the same scope line. (Note that one need not write down both conjuncts; the two possible uses of the rule are compressed in a single schematic representation.)

Conjunction Elimination

$\alpha \wedge \beta$	Already Derived
$\vdots$	
α	$\wedge$ E
$\vdots$	
β	$\wedge$ E

3.4 Disjunction Rules

The rule for introducing a disjunction ($\vee$ I) is very straightforward. One may prepend or append any sentence as a disjunct to any given sentence. (As with conjunction, only one application of the rule need be made at any one time.)

Disjunction Introduction

α	Already Derived
$\vdots$	
$\alpha \vee \beta$	$\vee$ I
$\vdots$	
$\beta \vee \alpha$	$\vee$ I

The rule for eliminating a disjunction, also known as "simple dilemma" and as "constructive dilemma" ($\vee$ E), is the most complicated of all the rules. If a disjunction occurs in a derivation, and one derives in the same scope line a single sentence by assuming each of the disjuncts, one may discharge the two assumptions and write down the derived sentence.

Disjunction Elimination

$\alpha \vee \beta$	Already Derived
$\quad \vert\ \alpha$	Assumption
$\quad \vert\ \vdots$	
$\quad \vert\ \gamma$	
$\quad \vert\ \beta$	Assumption
$\quad \vert\ \vdots$	
$\quad \vert\ \gamma$	
γ	$\vee$ E

3.5 Material Conditional Rules

The rule for introducing a material conditional (⊃ I) is moderately complex. One makes an assumption, derives a sentence from that assumption in the same scope line, discharges the assumption, and writes down a material conditional with the assumption as the antecedent and the derived sentence as the consequent.

Material Conditional Introduction

$$
\begin{array}{ll}
\quad \mid \; \alpha & \text{Assumption} \\
\quad \mid \; \vdots & \\
\quad \mid \; \beta & \\
\alpha \supset \beta & \supset \text{I}
\end{array}
$$

The elimination rule for the material conditional (⊃ E), widely known as "modus ponens," is quite simple. If one has derived a conditional and has also derived its antecedent within the same scope line, then one may write down the consequent of the conditional.

Material Conditional Elimination

$$
\begin{array}{ll}
\mid \; \alpha & \text{Already Derived} \\
\mid \; \vdots & \\
\mid \; \alpha \supset \beta & \\
\mid \; \vdots & \\
\mid \; \beta & \supset \text{E}
\end{array}
$$

It does not matter whether the conditional or its antecedent occurs first.

3.6 Material Biconditional Rules

Because the material biconditional is semantically equivalent to the conjunction of two material conditionals, the operator is introduced (using ≡ I) by using twice the same procedure that introduces conditionals. The sentence on one side is derived in the scope of the assumption of the sentence on the other side, and vice-versa.

Material Biconditional Introduction

$$
\begin{array}{ll}
\quad \mid \; \alpha & \text{Assumption} \\
\quad \mid \; \vdots & \\
\quad \mid \; \beta & \\
\quad \mid \; \beta & \text{Assumption} \\
\quad \mid \; \vdots & \\
\quad \mid \; \alpha & \\
\alpha \equiv \beta & \equiv \text{I}
\end{array}
$$

The elimination rule for the material biconditional (≡ E) likewise reflects the operator's relation to the conditional. If a biconditional is present and either sentence making it up is also present, then the other sentence can be written down.

Material Biconditional Elimination

α	Already Derived
$\vdots$	
$\alpha \equiv \beta$	
$\vdots$	
β	≡ E

β	Already Derived
$\vdots$	
$\alpha \equiv \beta$	
$\vdots$	
α	≡ E

It does not matter whether the biconditional or its component sentence occurs first.

3.7 *Falsum* Rules

The introduction rule for *falsum* (⊥ I) allows one to write down '⊥' when both α and $\sim\alpha$ (in either order) occur on the same scope line.

***Falsum* Introduction**

α	Already Derived
$\vdots$	
$\sim\alpha$	Already Derived
$\bot$	⊥ I

Falsum is "eliminated" (using ⊥ E) in the sense that when it occurs, any sentence whatsoever can be written down in the current scope line.

***Falsum* Elimination**

$\bot$	Already Derived
α	⊥ E

3.8 Derived Rules Using *Falsum*

Here we introducte two *derived rules* which are very useful in Predicate Logic derivations. Derived rules are rules that could always be dispensed with in favor of the primitive rules that have just been stated.

Negation Introduction-⊥

$$
\begin{array}{l|ll}
 & \alpha & \text{Assumption} \\ \cline{2-2}
 & \vdots & \\
 & \bot & \\
\multicolumn{2}{l}{\sim\alpha} & \sim \text{I-}\bot
\end{array}
$$

Negation Elimination-⊥

$$
\begin{array}{l|ll}
 & \sim\alpha & \text{Assumption} \\ \cline{2-2}
 & \vdots & \\
 & \bot & \\
\multicolumn{2}{l}{\alpha} & \sim \text{E-}\bot
\end{array}
$$

We can show that the derived rules hold by providing a template for generating the same results without their use. The illustration will be made for ~ I-⊥ only. Suppose that one can derive ⊥ from assumption α. We show that one can subsequently discharge the assumption and write $\sim\alpha$.

$$
\begin{array}{l|ll}
 & \alpha & \text{Assumption} \\ \cline{2-2}
 & \vdots & \\
 & \bot & \text{Derived from } \alpha \\
 & \beta & \bot\ \text{E} \\
 & \sim\beta & \bot\ \text{E} \\
\multicolumn{2}{l}{\sim\alpha} & \sim \text{I}
\end{array}
$$

3.9 Derivations of Schemata

The derivation rules are set up to allow derivations using sentences of *SL*. We will have occasion to use derivations whose steps contain instead meta-logical *schemata* of *SL* sentences. For example, the expression '$\alpha \wedge \beta$' is not a sentence of *SL*, but it shows schematically the form of *SL* sentences. Derivations of schemata will be given using the same annotations as for derivations using *SL* sentences.

3.10 Derivational Properties and Relations

As with the semantical system *SI*, the derivational system *SD* has various properties that are the subject of meta-theorems. The properties corresponding to Bivalence and Truth-Functionality cannot be understood until we have defined the relations and properties corresponding to semantical entailment (derivability) and validity (theoremhood). So we shall turn to them next.

3.10.1 Derivability

The *derivability relation*, $\{\gamma_1, \cdots, \gamma_n\} \vdash_{SD} \alpha$, holds between a set of sentences $\{\gamma_1, \cdots, \gamma_n\}$ and a sentence α when each member of the set is an assumption not in the scope of any other assumption, and α is a step in the immediate scope of those assumption(s). The derivability relation is the relation of logical consequence in the derivational system.

3.10.2 Derivational Equivalence

Two sentences α and β are *derivationally equivalent* if and only if $\{\alpha\} \vdash_{SD} \beta$ and $\{\beta\} \vdash_{SD} \alpha$. Note that derivational equivalence cannot be defined in the same manner as was semantical equivalence (having the same truth-value on all interpretations), since the sentences are not interpreted by the derivational system.

3.10.3 Theoremhood

A sentence α derived from no undischarged assumptions is said to be a *theorem* of *SD*. We signify the theoremhood of α by writing '$\vdash_{SD} \alpha$.' We will here give derivations that prove the theoremhood of the three schemata for the axiomatic system *SA* described below in 5.1.

Proof that $\vdash_{SD} \alpha \supset (\beta \supset \alpha)$

1	$\mid \alpha$	Assumption
2	$\mid \mid \beta$	Assumption
3	$\mid \mid \alpha$	1 R
4	$\mid \beta \supset \alpha$	2-3 ⊃ I
5	$\alpha \supset (\beta \supset \alpha)$	1-4 ⊃ I

Proof that $\vdash_{SD} (\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

1	$\mid \alpha \supset (\beta \supset \gamma)$	Assumption
2	$\mid \mid \alpha \supset \beta$	Assumption
3	$\mid \mid \alpha \supset (\beta \supset \gamma)$	1 R
4	$\mid \mid \mid \alpha$	Assumption
5	$\mid \mid \mid \alpha \supset (\beta \supset \gamma)$	3 R
6	$\mid \mid \mid \beta \supset \gamma$	4 5 ⊃ E
7	$\mid \mid \mid \alpha \supset \beta$	2 R
8	$\mid \mid \mid \beta$	4 7 ⊃ E
9	$\mid \mid \mid \gamma$	6 8 ⊃ E
10	$\mid \mid \alpha \supset \gamma$	4-9 ⊃ I
11	$\mid (\alpha \supset \beta) \supset (\alpha \supset \gamma)$	2-10 ⊃ I
12	$(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$	1-11 ⊃ I

Proof that $\vdash_{SD} (\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset B)$

1	$\sim\beta \supset \sim\alpha$	Assumption
2	$\sim\beta \supset \alpha$	Assumption
3	$\sim\beta \supset \sim\alpha$	1 R
4	$\sim\beta$	1 Assumption
5	$\sim\beta \supset \alpha$	2 R
6	$\sim\beta \supset \sim\alpha$	3 R
7	$\sim\alpha$	4 5 ⊃ E
8	α	4 6 ⊃ E
9	β	3-8 ~ E
10	$(\sim\beta \supset \alpha) \supset \beta$	2-9 ⊃ I
11	$(\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)$	1-10 ⊃ I

3.11 Excluded Middle

Corresponding to the semantical property of Bivalence described in Section 2.3.1 above is a sentence-schema known as the *Excluded Middle*, $\alpha \vee \sim\alpha$ is a theorem of *SD*. The derivational system does not have a way of expressing falsehood directly. Intuitively, we take every step of a derivation to be an assertion of truth. Falsehood can, however, be expressed indirectly. To assert $\sim\alpha$ is, from the point of view of *SI*, tantamount to asserting that α is false. So the semantical property of sentences, that each one is either true or false, is expressed in *SL* sentences of the form $\alpha \vee \sim\alpha$. It may be derived as follows.

1	$\sim(\alpha \vee \sim\alpha)$	Assumption
2	α	Assumption
3	$\alpha \vee \sim\alpha$	2 ∨ I
4	$\sim(\alpha \vee \sim\alpha)$	1 R
5	$\sim\alpha$	2-4 ~ I
6	$\alpha \vee \sim\alpha$	5 ∨ I
7	$\alpha \vee \sim\alpha$	1-6 ~ E

3.12 Non-Contradiction

The counterpart to Truth-Functionality described in Section 2.3.2 above is the sentence-schema known as *Non-Contradiction*, $\sim(\alpha \wedge \sim\alpha)$, which is a theorem-schema of *SD*. Once again, reference to falsehood is made through negation. From the standpoint of *SI*, sentences of this sort assert that it is not the case that α and $\sim\alpha$ are both true, in which case α is not both true and false.

1	$\alpha \wedge \sim\alpha$	Assumption
2	α	1 $\wedge$ E
3	$\sim\alpha$	1 $\wedge$ E
4	$\sim(\alpha \wedge \sim\alpha)$	1-3 $\sim$ I

The reader will not fail to notice how easy these derivations are compared with the semantical derivations made in the previous section.

3.13 Derivational Consistency

Corresponding to Semantical Consistency described in Section 2.3.6 above is the notion of *derivational consistency*. A set of sentences Γ of *SL* is derivationally consistent (d-consistent), if and only if it is not the case that there is a sentence α of *SL* such that $\Gamma \vdash_{SD} \alpha$ and $\Gamma \vdash_{SD} \sim\alpha$. Any set of sentences which is not d-consistent is *d-inconsistent*. We can prove the following meta-theorem:

$\Gamma \vdash_{SD} \alpha$ if and only if $\Gamma \cup \sim\alpha$ is d-inconsistent.

Suppose $\Gamma \vdash_{SD} \alpha$. Then there is a derivation of α from the members of Γ as assumptions. Adding an assumption does not affect the proof, so $\Gamma \cup \sim\alpha \vdash_{SD} \alpha$. By the rule of Reiteration, $\Gamma \cup \sim\alpha \vdash_{SD} \sim\alpha$. Therefore, $\Gamma \cup \sim\alpha$ is d-inconsistent.

This reasoning can be illustrated by the following derivation-schemata. First, suppose that $\Gamma \vdash_{SD} \alpha$.

1	γ_1	Assumption
2	...	
3	γ_n	Assumption
4	...	
5	α	By Hypothesis

We add $\sim\alpha$ as a further assumption.

1	γ_1	Assumption
2	...	
3	γ_n	Assumption
4	$\sim\alpha$	Assumption
5	...	
6	α	By Hypothesis

We derive $\sim\alpha$ from $\Gamma \cup \sim\alpha$.

1	γ_1	Assumption
2	$\dots$	
3	γ_n	Assumption
4	$\sim\alpha$	Assumption
5	$\dots$	
6	$\sim\alpha$	Reiteration

Now suppose $\Gamma \cup \sim\alpha$ is d-inconsistent. Suppose further that the members of Γ are the outermost assumptions in a derivation. Assume $\sim\alpha$ and reiterate all the members of Γ. Because $\Gamma \cup \sim\alpha$ is d-inconsistent, we may then derive a pair of sentences β and $\sim\beta$. The assumption of $\sim\alpha$ may be discharged and α written down as the last step in the derivation, in which case $\Gamma \vdash_{SD} \alpha$.

1	γ_1	Assumption
2	$\dots$	
3	γ_n	Assumption
4	$\sim\alpha$	Assumption
5	γ_1	Reiteration
6	$\dots$	
7	γ_n	Reiteration
8	$\dots$	
9	β	From d-inconsistency
10	$\dots$	
11	$\sim\beta$	From d-inconsistency $\Gamma \cup \sim\alpha$
12	α	$\sim$ E

The reader will note that there is a structural similarity between this result and the corresponding result for semantical inconsistency.

4 Soundness and Completeness

Thus far, our meta-logical results have been confined to either the semantical system *SI* or the derivational system *SD*. There are further meta-theorems which concern the relation between the two systems. The first is the *soundness* of *SD*. A derivational system is sound relative to a semantical system just in case every relation of derivability is a relation of semantical entailment. With respect to our present systems, if $\{\gamma_1, \dots, \gamma_n\} \vdash_{SD} \alpha$, then $\{\gamma_1, \cdots, \gamma_n\} \vDash_{SI} \alpha$. A derivational system is *complete* relative to a semantical system just in case all semantical entailments are derivable. Applied to Sentential Logic, this means that if $\{\gamma_1, \cdots, \gamma_n\} \vDash_{SI} \alpha$, then $\{\gamma_1, \dots, \gamma_n\} \vdash_{SD} \alpha$. Because theoremhood and validity are degenerate cases of derivability and semantic entailment, respectively, we can define more restricted (or "weaker") senses of soundness and completeness. If *SD* is sound relative to *SI*, then $\vdash_{SD} \alpha$, then $\vDash_{SI} \alpha$. If *SD* is complete relative to *SI*, then if $\vDash_{SI} \alpha$, then $\vdash_{SD} \alpha$.

4.1 Soundness

We may think of the relation of semantical entailment as one which excludes the possibility of a set of sentences being true and its consequence being false. If a set of derivation rules is such that all derivations are semantical entailments, then the derivation rules do not allow the inference of a false consequence from true premises. This accords with an intuitive sense in which the system of derivation rules is sound.

The soundness of *SD* relative to *SI* will not be proved here, but we can give an indication of how the proof goes. The main work of the proof is to establish a lemma: at any point in a derivation, if all the undischarged assumptions $\{\gamma_1, \ldots, \gamma_n\}$ up to that point are true on an interpretation, then the last formula α is true on that interpretation as well.[34] The lemma applies to the limiting case where α is in the immediate scope of the outermost assumptions $\{\gamma_1, \ldots, \gamma_n\}$, in which case $\{\gamma_1, \ldots, \gamma_n\} \vdash_{SD} \alpha$.

The lemma is proved by mathematical induction on the length of the derivation. Suppose *n* steps of a derivation have been constructed. The proof shows that if the sentences needed for the application of the rule at the *n*th step are true given that the undischarged assumptions are true, then the result of the application of the rule is true as well. As an illustration, consider Conjunction Elimination. Suppose that $\{\gamma_1, \ldots, \gamma_n\}$ are all the undischarged assumptions, and that $\alpha \wedge \beta$ is a sentence that lies in their scope. By the inductive hypothesis, if $\{\gamma_1, \ldots, \gamma_n\}$ are all true, then $\alpha \wedge \beta$ is true. But we have shown already that if this is the case, then α is true and β is true. So, if $\{\gamma_1, \ldots, \gamma_n\}$ are all true, so are α and β. Reasoning for the rules involving assumptions is more complicated and will not be described here.

Another way to look at soundness is by noticing the parallel between the derivational rules and the semantical reasoning used to establish semantical entailment. Here we assert without proof that each of the rules of *SD* can be mapped onto a corresponding argument that uses only semantical rules. For example, the rule of Conjunction Elimination corresponds to the rule that if the value of a conjunction is **T**, then the values of each of its conjuncts are **T**.

Conjunction Elimination

$\alpha \wedge \beta$	Already Derived
$\ldots$	
α	$\wedge$ E

Semantical proof that: $\alpha \wedge \beta \vDash_{SI} \alpha$

$\mathbf{v_I}(\alpha \wedge \beta) = \mathbf{T}$	Assumption
$\ldots$	
$\mathbf{v_I}(\alpha \wedge \beta) = \mathbf{T} \rightarrow \mathbf{v_I}(\alpha) = \mathbf{T}$	**SR-**$\wedge$
$\mathbf{v_I}(\alpha) = \mathbf{T}$	Modus Ponens

We can think of the derivational rule as mirroring the use of semantical rules, but in a "cleaner" or "less-cluttered" way. The derivational system dispenses with references to truth-values because it implicitly takes all the truth-values to be **T**. Falsehood of a sentence α is displayed by the using its negation, $\sim\alpha$. Here is another example to illustrate the point.

[34] A *lemma* is is a principal result used in proving a meta-theorem.

Negation Introduction

$$\begin{array}{l|ll} & \alpha & \text{Assumption} \\ & \ldots & \\ & \beta & \\ & \ldots & \\ & {\sim}\beta & \\ \multicolumn{2}{l}{{\sim}\alpha} & {\sim}\ \text{I} \end{array}$$

Semantical proof of the Soundness of ~ Introduction

$$\begin{array}{l|ll} & \mathbf{v_I}(\alpha) = \mathbf{T} & \text{Assumption} \\ & \ldots & \\ & \mathbf{v_I}(\beta) = \mathbf{T} & \text{Semantical Reasoning} \\ & \ldots & \\ & \mathbf{v_I}({\sim}\,\beta) = \mathbf{T} & \text{Semantical Reasoning} \\ & \mathbf{v_I}(\beta) = \mathbf{F} & \mathbf{SR\text{-}{\sim}} \\ & \mathbf{v_I}(\beta) = \mathbf{T} \bigwedge \mathbf{v_I}(\beta) = \mathbf{F} & \bigwedge \text{I} \\ & \neg(\mathbf{v_I}(\beta) = \mathbf{T} \bigwedge \mathbf{v_I}(\beta) = \mathbf{F}) & \mathbf{TF} \\ \multicolumn{2}{l}{\neg\mathbf{v_I}(\alpha) = \mathbf{T}} & \neg\ \text{I} \\ \multicolumn{2}{l}{\mathbf{v_I}(\alpha) = \mathbf{F}} & \mathbf{BV} \\ \multicolumn{2}{l}{\mathbf{v_I}({\sim}\alpha) = \mathbf{T}} & \mathbf{SR\text{-}{\sim}} \end{array}$$

We will exploit the parallelism between semantical rules and derivational rules when we construct derivational rules for systems of modal logic.

4.2 Completeness

If we think of the relation of semantical entailment in *SI* as exhaustive of the logical consequences holding among sentences of *SL*, then the derivational system is, intuitively, complete if it can reproduce all those consequences. Proofs of completeness are a good deal more difficult than proofs of soundness.

There are many strategies for completeness proofs. The most intuitively satisfying proofs are *constructive*, showing how for any semantical entailment a derivation can be constructed. These proofs are complicated in detail, and will be discussed further below.

Another proof-strategy is *non-constructive*. In such proofs, all that is shown is that for every semantical entailment there must be a derivation, but what that derivation looks like is completely unspecified. This type of proof was first advanced by Leon Henkin, and so it has come to be known as the "Henkin-style" proof-strategy for completeness.[35] One advantage of the Henkin approach is that it is easily adapted to modal logics.

[35] "The Completeness of the First-Order Functional Calculus," *Journal of Symbolic Logic* Vol. 14 (1949), pp. 42-50.

Here we will only sketch the strategy for proving completeness in the manner of Henkin. The key move is to show how to begin with a consistent set of sentences and then build a "maximal" d-consistent set of sentences Γ', of which the original set is a subset.[36] "Lindenbaum's Lemma," which states that every d-consistent set is a subset of a maximal d-consistent set is then proved, using the specific characteristics of *SD* derivations, which determine what is and what is not a member of a maximal d-consistent set.

It is next shown that there is an interpretation according to which a given sentence is a member of Γ' if and only if that sentence has the value **T** on that interpretation.[37] This is done by using a special recipe for constructing an interpretation which guarantees that all and only the members of Γ' have the value **T**, and hence that Γ' is s-consistent. The recipe is easy: assign to all the sentence letters in Γ' the value **T** and to all the sentence letters not in Γ' the value **F**. It is then proved by mathematical induction that on such an interpretation, all and only members of Γ' have the value **T**. This proof also relies on the specific features of *SD* derivations, which determine what is and what is not a member of Γ'. Finally, if Γ' is such that there is an interpretation on which all its sentences have the value *T*, then all its subsets have the value *T* as well, by simple reasoning in predicate logic.

We sketch the proof-strategy in the form of a meta-logical derivation.

Sketch of Proof that: If Γ is d-consistent, then Γ is s-consistent.

1	Γ is d-consistent	Assumption
2	$\Gamma \subseteq$ a maximal d-consistent set Γ'	1 "Lindenbaum's Lemma"
3	$(\Sigma\mathbf{I})(\Pi\alpha)(\alpha \in \Gamma' \leftrightarrow \mathbf{v_I}(\alpha) = T)$	By Mathematical Induction
4	Γ is s-consistent	3 Definition of 's-consistent'
5	Γ is d-consistent $\rightarrow$ Γ is s-consistent	1-4 $\rightarrow$ I

Now we may apply this strategy to a set $\Gamma \cup {\sim}\alpha$, which we assume to be d-consistent. By Lindenbaum's Lemma, it is a subset of a maximal d-consistent set Γ'. We apply the interpretation that makes all the sentences in Γ' have the value *T*, in which case all the sentences in $\Gamma \cup {\sim}\alpha$ have the value **T**; that is, $\Gamma \cup {\sim}\alpha$ is s-consistent.

Sketch of Proof that: If $\Gamma \vDash_{SI} \alpha$, then $\Gamma \vdash_{SD} \alpha$.

1	$\Gamma \vDash_{SI} \alpha$	Assumption
2	$\Gamma \vDash_{SI} \alpha \rightarrow \Gamma \cup {\sim}\alpha$ is s-inconsistent	Proved in 2.3.6
3	$\Gamma \cup {\sim}\alpha$ is s-inconsistent	1 2 $\rightarrow$ E
4	$\Gamma \cup {\sim}\alpha$ is d-consistent $\rightarrow$ $\Gamma \cup {\sim}\alpha$ is s-consistent	Instance of above result
5	$\Gamma \cup {\sim}\alpha$ is d-inconsistent	3 4 Modus Tollens
6	$\Gamma \cup {\sim}\alpha$ is d-inconsistent $\rightarrow \Gamma \vdash_{SD} \alpha$	Proved in 3.13
7	$\Gamma \vdash_{SD} \alpha$	5 6 $\rightarrow$ E

This general strategy can be adapted to any pair of systems for which d-inconsistency and s-inconsistency can be defined in the way it has been defined here. This will include the modal logics that will be studied

[36] We use '$\subseteq$' to denote the subset relation: $\alpha \subseteq \beta$ if and only if every member of α is a member of β.

[37] For arbitrary sentences α and β, we write '$\alpha \in \beta$' to indicate the membership of α in β.

in later modules. The means of implementing the strategy in the proof of Lindenbaum's Lemma and of the equivalence between the s-consistency and d-consistency of maximal will depend on the specific character of the systems studied.

The un-intuitive character of this mode of proof is apparent. A more intuitively satisfying approach would be to exploit the striking parallelism between semantical reasoning and derivations, as was done with soundness. However, there is a difficulty in so doing. It is not the case that any semantical derivation can be converted into a derivation in *SD* by making the appropriate conversion from sentences having the value *F* to the negation of those sentences. Here is an earlier example of semantical reasoning, followed by a corresponding derivation.

Semantical proof that: $\vDash_{SI} \alpha \vee {\sim}\alpha$

1	$\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$	**BV**
2	$\quad\mid\ \mathbf{v_I}(\alpha) = \mathbf{T}$	Assumption
3	$\quad\mid\ \mathbf{v_I}(\alpha) = \mathbf{T} \rightarrow \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	**SR-**$\vee$
4	$\quad\mid\ \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	2 3 $\rightarrow$ I
5	$\quad\mid\ \mathbf{v_I}(\alpha) = \mathbf{F}$	Assumption
6	$\quad\mid\ \mathbf{v_I}(\alpha) = \mathbf{F} \rightarrow \mathbf{v_I}({\sim}\alpha) = \mathbf{T}$	**SR-**$\sim$
7	$\quad\mid\ \mathbf{v_I}({\sim}\alpha) = \mathbf{T}$	5 6 $\rightarrow$ E
8	$\quad\mid\ \mathbf{v_I}({\sim}\alpha) = \mathbf{T} \rightarrow \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	**SR-**$\vee$
9	$\quad\mid\ \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	7 8 $\rightarrow$ I
10	$\mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	1 2-4 5-9 $\bigvee$ E

There is nothing in the derivational system that corresponds to the assertion of the rule of Bivalence. The closest we can come would be the formula $\alpha \vee {\sim}\alpha$, which is the very thing we are trying to derive. The derivation of that schema in *SD* proceeds as follows.

Proof that: $\vdash_{SD} \alpha \vee {\sim}\alpha$

1	$\mid\ {\sim}(\alpha \vee {\sim}\alpha)$	Assumption
2	$\mid\ \mid\ \alpha$	Assumption
3	$\mid\ \mid\ \alpha \vee {\sim}\alpha$	2 $\vee$ I
4	$\mid\ \mid\ {\sim}(\alpha \vee {\sim}\alpha)$	1 R
5	$\mid\ {\sim}\alpha$	2-4 $\sim$ I
6	$\mid\ \alpha \vee {\sim}\alpha$	5 $\vee$ I
7	$\alpha \vee {\sim}\alpha$	1-6 $\sim$ E

Now this derivation can be seen as a conversion of a meta-logical semantical derivation.

Another semantical proof that: $\vDash_{SI} \alpha \vee {\sim}\alpha$

1	$\neg(\mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T})$	Assumption
2	$\quad \mathbf{v_I}(\alpha) = \mathbf{T}$	Assumption
3	$\quad \mathbf{v_I}(\alpha) = \mathbf{T} \rightarrow \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	**SR-**$\vee$
4	$\quad \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	2 3 $\rightarrow$ E
5	$\quad \neg(\mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T})$	1 Reiteration
6	$\neg\mathbf{v_I}(\alpha) = \mathbf{T}$	2-5 $\neg$ I
7	$\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$	**BV**
8	$\mathbf{v_I}(\alpha) = \mathbf{F}$	6 7 Disjunctive Syllogism
9	$\mathbf{v_I}(\alpha) = \mathbf{F} \rightarrow \mathbf{v_I}({\sim}\alpha) = \mathbf{T}$	**SR-**$\sim$
10	$\mathbf{v_I}({\sim}\alpha) = \mathbf{T}$	8 9 $\rightarrow$ E
11	$\mathbf{v_I}({\sim}\alpha) = \mathbf{T} \rightarrow \mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	**SR-**$\vee$
12	$\mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	10 11 $\rightarrow$ E
13	$\mathbf{v_I}(\alpha \vee {\sim}\alpha) = \mathbf{T}$	1-12 $\neg$ E

To try to show completeness by exploiting the parallism between the two pieces of reasoning, we would have to show how to set up all our semantical derivations in a way that would allow them to be converted to derivations in *SD*. There may be a way of making the derivations proving semantical entailment conform precisely to how a truth-table is constructed. This is a task which will not be undertaken here.

5 Axiomatic Formulation of Sentential Logic

The earliest formulations of sentential logic were axiomatic. In an axiom system, there is a set of sentences called *axioms* and a set of rules of inference are specified. A *theorem* is defined as an axiom or what follows from axioms by way of rules of inference. The primary aim of axiomatic systems is to generate an acceptable list of theorems.

5.1 The Axiom System *SA*

The following is one among many possible axiomatizations of Sentential Logic. This version (which we will call "*SA*") describes an axiom system using *axiom schemata*. Rather than specifying axioms, the system shows through the schemata the form that the axioms have. The axioms themselves are sentences of *SL* with the displayed form. Since there are infinitely many sentences of *SL*, there are infinitely many axioms in this system.[38]

Axioms

SA 1 $\vdash_{SA} \alpha \supset (\beta \supset \alpha)$

[38]Alternatively, one could specify three axioms (for this formulation) using sentences of *SL* and add a rule of "uniform substitution" which generates theorems by substituting other sentences for those in the axioms.

SA 2 $\vdash_{SA} (\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

SA 3 $\vdash_{SA} ({\sim}\beta \supset {\sim}\alpha) \supset (({\sim}\beta \supset \alpha) \supset \beta)$

Rule of Inference

Modus Ponens From $\vdash_{SA} \alpha$ and $\vdash_{SA} \alpha \supset \beta$, infer $\vdash_{SA} \beta$.

None of the axioms of *SA* contains any of the operators '∧,' '∨,' or '≡.' These operators may be defined contextually.

$\alpha \wedge \beta =_{\mathrm{Df}} {\sim}(\alpha \supset {\sim}\beta)$

$\alpha \vee \beta =_{\mathrm{Df}} {\sim}\alpha \supset \beta$

$\alpha \equiv \beta =_{\mathrm{Df}} {\sim}((\alpha \supset \beta) \supset {\sim}(\beta \supset \alpha))$

Theorems of *SA* may be converted to definitionally equivalent theorems by making the appropriate substitutions. For example, $\vdash_{SA} {\sim}\alpha \supset {\sim}\alpha$. Based on the definition (and Double Negation, which holds in *SA*), we may assert that $\vdash_{SA} \alpha \vee {\sim}\alpha$.

The following is an example of an axiomatic proof that $\vdash_{SA} \alpha \supset \alpha$. (It is easily adapted to prove $\vdash_{SA} {\sim}\alpha \supset {\sim}\alpha$.) The theorem is very simple, yet the proof is quite cumbersome.

Axiomatic Proof that: $\vdash_{SA} \alpha \supset \alpha$

1	$(\alpha \supset ((\alpha \supset \alpha) \supset \alpha)) \supset ((\alpha \supset (\alpha \supset \alpha)) \supset (\alpha \supset \alpha))$	**SA 2**
2	$\alpha \supset ((\alpha \supset \alpha) \supset \alpha)$	**SA 1**
3	$(\alpha \supset (\alpha \supset \alpha)) \supset (\alpha \supset \alpha)$	1 2 Modus Ponens
4	$\alpha \supset (\alpha \supset \alpha)$	**SA 1**
5	$\alpha \supset \alpha$	3 4 Modus Ponens

The chief advantage of the axiomatic formulation of a system for *SL* is that it facilitates proofs of soundness, completeness, and other meta-logical properties and relations. Here, we will use the axiom system to illustrate the way in which semantical reasoning and derivations work in the context of Sentential Logic.

5.2 *SA* and *SI*

The axiom system *SA* can be proved to be sound and complete relative to the semantical system *SI*. Here we will only prove the weak form of soundness, that all theorems of *SA* are valid in *SI*.

The proof is by mathematical induction on the number n of steps in the proof of the theoremhood of an arbitrary formula α.

Basis Step: $n = 1$.

If there is only one step in the proof of the theoremhood of α, then α is an axiom of *SA*. So we will prove that any axiom schema of *SA* has as instances only sentences valid in *SI*.

Semantical proof of Validity of SA 1: $\vDash_{SA} \alpha \supset (\beta \supset \alpha)$

1	$\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$	**BV**
2	$\quad \mathbf{v_I}(\alpha) = \mathbf{T}$	Assumption
3	$\quad \mathbf{v_I}(\alpha) = \mathbf{T} \rightarrow \mathbf{v_I}(\beta \supset \alpha) = \mathbf{T}$	**SR-⊃**
4	$\quad \mathbf{v_I}(\beta \supset \alpha) = \mathbf{T}$	2 3 → E
5	$\quad \mathbf{v_I}(\beta \supset \alpha) = \mathbf{T} \rightarrow \mathbf{v_I}(\alpha \supset (\beta \supset \alpha)) = \mathbf{T}$	**SR-⊃**
6	$\quad \mathbf{v_I}(\alpha \supset (\beta \supset \alpha)) = \mathbf{T}$	4 5 → E
7	$\quad \mathbf{v_I}(\alpha) = \mathbf{F}$	Assumption
8	$\quad \mathbf{v_I}(\alpha) = \mathbf{F} \rightarrow \mathbf{v_I}(\alpha \supset (\beta \supset \alpha)) = \mathbf{T}$	**SR-⊃**
9	$\quad \mathbf{v_I}(\alpha \supset (\beta \supset \alpha)) = \mathbf{T}$	7 8 → E
10	$\mathbf{v_I}(\alpha \supset (\beta \supset \alpha)) = \mathbf{T}$	1 2-6 7-9 ⋁ E

The moves made in the proof can be illustrated through truth-tables. Only the values needed to get the result are listed.

α	$\beta \supset \alpha$	$\alpha \supset (\beta \supset \alpha)$
T	**T**	**T**

α	$\alpha \supset (\beta \supset \alpha)$
F	**T**

Semantical proof of Validity of SA 2: $\vDash_{SA} (\alpha \supset (\beta \supset \gamma) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

1	$\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$	**BV**
2	$\mathbf{v_I}(\alpha) = \mathbf{T}$	Assumption
3	$\mathbf{v_I}(\gamma) = \mathbf{T} \bigvee \mathbf{v_I}(\gamma) = \mathbf{F}$	**BV**
4	$\mathbf{v_I}(\gamma) = \mathbf{T}$	Assumption
5	$\mathbf{v_I}(\alpha \supset \gamma) = \mathbf{T}$	4 **SR-**$\supset$
6	$\mathbf{v_I}((\alpha \supset \beta) \supset (\alpha \supset \gamma)) = \mathbf{T}$	5 **SR-**$\supset$
7	$\mathbf{v_I}((\alpha \supset (\beta \supset \gamma)) \supset (\alpha \supset \beta) \supset (\alpha \supset \gamma)) = \mathbf{T}$	6 **SR-**$\supset$
8	$\mathbf{v_I}(\gamma) = \mathbf{F}$	Assumption
9	$\mathbf{v_I}(\beta) = \mathbf{T} \bigvee \mathbf{v_I}(\beta) = \mathbf{F}$	**BV**
10	$\mathbf{v_I}(\beta) = \mathbf{T}$	Assumption
11	$\mathbf{v_I}(\beta \supset \gamma) = \mathbf{F}$	8 10 **SR-**$\supset$
12	$\mathbf{v_I}(\alpha \supset (\beta \supset \gamma)) = \mathbf{F}$	11 **SR-**$\supset$
13	$\mathbf{v_I}((\alpha \supset (\beta \supset \gamma)) \supset (\alpha \supset (\beta \supset \gamma))) = \mathbf{T}$	12 **SR-**$\supset$
14	$\mathbf{v_I}(\beta) = \mathbf{F}$	Assumption
15	$\mathbf{v_I}(\alpha \supset \beta) = \mathbf{F}$	2 14 **SR-**$\supset$
16	$\mathbf{v_I}((\alpha \supset \beta) \supset (\alpha \supset \gamma)) = \mathbf{T}$	15 **SR-**$\supset$
17	$\mathbf{v_I}((\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))) = \mathbf{T}$	16 **SR-**$\supset$
18	$\mathbf{v_I}((\alpha \supset (\beta \supset \gamma)) \supset (\alpha \supset \beta) \supset (\alpha \supset \gamma) = \mathbf{T}$	9 10-13 14-17 $\bigvee$ E
19	$\mathbf{v_I}((\alpha \supset (\beta \supset \gamma)) \supset (\alpha \supset \beta) \supset (\alpha \supset \gamma) = \mathbf{T}$	3 4-7 8-18$\bigvee$ E
20	$\mathbf{v_I}(\alpha) = \mathbf{F}$	Assumption
21	$\mathbf{v_I}(\alpha \supset \gamma) = \mathbf{T}$	20 **SR-**$\supset$
22	$\mathbf{v_I}((\alpha \supset \beta) \supset (\alpha \supset \gamma)) = \mathbf{T}$	21 **SR-**$\supset$
23	$\mathbf{v_I}((\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))) = \mathbf{T}$	22 **SR-**$\supset$
24	$\mathbf{v_I}((\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))) = \mathbf{T}$	1 2-19 20-23 $\bigvee$ E

Again, we illustrate the moves in the proof using truth-tables.

α	γ	$\alpha \supset \gamma$	$(\alpha \supset \beta) \supset (\alpha \supset \gamma)$	$(\alpha \supset (\beta \supset \gamma) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
T	**T**	**T**	**T**	**T**

α	β	γ	$\beta \supset \gamma$	$\alpha \supset (\beta \supset \gamma)$	$(\alpha \supset (\beta \supset \gamma) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
T	**T**	**F**	**F**	**F**	**T**

α	β	$\alpha \supset \beta$	$(\alpha \supset \beta) \supset (\alpha \supset \gamma)$	$(\alpha \supset (\beta \supset \gamma) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
T	**F**	**F**	**T**	**T**

α	$\alpha \supset \gamma$	$(\alpha \supset \beta) \supset (\alpha \supset \gamma)$	$(\alpha \supset (\beta \supset \gamma) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$
F	**T**	**T**	**T**

Semantical proof of Validity of SA 3: $\vDash_{SA} (\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)$

1	$\mathbf{v_I}(\beta) = \mathbf{T} \bigvee \mathbf{v_I}(\beta) = \mathbf{F}$	**BV**
2	$\mathbf{v_I}(\beta) = \mathbf{T}$	Assumption
3	$\mathbf{v_I}(((\sim\beta \supset \alpha) \supset \beta) = \mathbf{T}$	2 **SR-⊃**
4	$\mathbf{v_I}((\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)) = \mathbf{T}$	3 **SR-⊃**
5	$\mathbf{v_I}(\beta) = \mathbf{F}$	Assumption
6	$\mathbf{v_I}(\sim\beta) = \mathbf{T}$	5 **SR-~**
7	$\mathbf{v_I}(\alpha) = \mathbf{T} \bigvee \mathbf{v_I}(\alpha) = \mathbf{F}$	**BV**
8	$\mathbf{v_I}(\alpha) = \mathbf{T}$	Assumption
9	$\mathbf{v_I}(\sim\alpha) = \mathbf{F}$	8 **SR-~**
10	$\mathbf{v_I}(\sim\beta \supset \sim\alpha) = \mathbf{F}$	6 9 **SR-⊃**
11	$\mathbf{v_I}((\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)) = \mathbf{T}$	10 **SR-⊃**
12	$\mathbf{v_I}(\alpha) = \mathbf{F}$	Assumption
13	$\mathbf{v_I}(\sim\beta \supset \alpha) = \mathbf{F}$	12 **SR-⊃**
14	$\mathbf{v_I}((\sim\beta \supset \alpha) \supset \beta) = \mathbf{T}$	13 **SR-⊃**
15	$\mathbf{v_I}((\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)) = \mathbf{T}$	14 **SR-⊃**
16	$\mathbf{v_I}((\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)) = \mathbf{T}$	7 8-11 12-15 ⋁ E
17	$\mathbf{v_I}((\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)) = \mathbf{T}$	1 2-4 5-16 ⋁ E

Again, the reasoning is illustrated with truth-tables.

β	$(\sim\beta \supset \alpha) \supset \beta$	$(\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)$
T	**T**	**T**

α	β	$\sim\alpha$	$\sim\beta$	$\sim\beta \supset \sim\alpha$	$(\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)$
T	**F**	**F**	**T**	**F**	**T**

α	β	$\sim\beta$	$\sim\beta \supset \alpha$	$(\sim\beta \supset \alpha) \supset \beta$	$(\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta)$
F	**F**	**T**	**F**	**T**	**T**

Inductive Hypothesis. Suppose that the sentences proved in all steps before n are valid. It will be shown that the sentence produced by step n itself is valid.

Induction Step. Because we have only one inference rule, Modus Ponens, all that is required, given the Inductive Hypothesis, is to show that if $\vDash_{SI} \alpha$ and $\vDash_{SI} \alpha \supset \beta$, then $\vDash_{SI} \beta$. Suppose this is the case. Then α has the value **T** on all interpretations, as does $\alpha \supset \beta$. By **SR-⊃**, it follows by easy reasoning that β has the value **T** on all interpretations, i.e., $\vDash_{SI} \alpha$.

This completes the proof of the Induction Step, and we can assert that no matter what the length of the proof of its theoremhood, if α is a theorem, then α is valid.

5.3 *SA* and *SD*

Earlier, we derived all three axioms of *SA* as theorems of *SD*. So we can say that all axioms of *SA* are theorems of *SD*. Further, the use of the Rule of Modus Ponens can be simulated in a derivation of the following form. Given that $\vdash \alpha$ and $\vdash \alpha \supset \beta$, α and $\alpha \supset \beta$ can be derived from no assumptions. We can then use $\supset$ Elimination to derive β. Since the two steps used to derive β do not depend on any assumptions, β does not either.

α	Derived from no assumptions
...	
$\alpha \supset \beta$	Derived from no assumptions
...	
β	$\supset$ E (Derived from no assumptions)

These results together show that the derivation system contains the axiom system. That is, anything that can be proved using the axioms and Modus Ponens can be proved using natural deduction. Proving the converse, that the axiom system contains the derivation system, is more difficult. One would have to show that any theorem produced by natural deduction can be proved axiomatically. This involves giving a method to convert derivations to axiomatic proofs.[39]

A final consideration is the notion of logical consequence, as opposed to mere theoremhood, in an axiom system. We can define a notion of deducibility in the system in the following way. Suppose Γ is a set of sentences of *SL* and α is a sentence of *SL*. A deduction of α from Γ is a sequence of sentences, with α as the last member, each of which is either an axiom of *SA*, a member of Γ, or the result of the use of Modus Ponens on two previous steps.

This concludes our survey of non-modal Sentential Logic. We are now ready to move on to study Modal Sentential Logic.

[39] See, for example, Richmond H. Thomason, *Symbolic Logic: An Introduction*, Chapter V.