Module 7
D and Equivalent Systems

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Contents

In this and subsequent modules, a number of systems stronger than the *K*-systems will be developed by adding restrictions to the accessibility relation, new derivational rules, and new axioms. Requiring that various conditions on accessibility hold means in general that more worlds are accessible to a given world, which removes obstacles to entailment, validity, etc. by limiting the range of potential counter-examples. Adding new rules of inference increases the range of derivations that can be made. And adding new axioms increases the stock of theorems that can be proved.

The weakest extensions of the *K*-systems are the *D*-systems. The semantical system *DI* will be examined first, followed by the derivational system *DD*. The axiom system *D* will be given a brief treatment. Following this, there will be a discussion of the suitablity of the *D*-systems for the various applications discussed in the last module.

1 The Semantical System *DI*

The semantical system *DI* is just like the system *KI* except for the requirement that **R** be *serial*. That is, each world must have at least one world accessible to it. We may express the serial character of a relation **R** as follows:

R is **serial** if and only if $(\Pi \mathbf{x})(\Sigma \mathbf{y})\mathbf{Rxy}$.

We can define a *DI*-frame as a set $\langle \mathbf{W}, \mathbf{R} \rangle$, such that :

$(\Pi \mathbf{w})(\mathbf{w} \in \mathbf{W} \rightarrow (\Sigma \mathbf{w}_i)(\mathbf{w}_i \in \mathbf{W} \wedge \mathbf{Rww}_i))$.[1]

An interpretation **I** is based on a frame, and the frame's set of worlds **W** is a member of **I**. So we can expand the description of a serial accessibility relation in a frame to that of a serial accessibility relation in an interpretation:

[1]The '$\in$' symbol indicates membership in a set.

$(\Pi \mathbf{w})((\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{W} \in \mathbf{I}) \to (\Sigma \mathbf{w}_i)((\mathbf{w}_i \in \mathbf{W} \bigwedge \mathbf{W} \in \mathbf{I}) \bigwedge \mathbf{Rww}_i))$.

Because the conjunction $\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{W} \in \mathbf{I}$ is somewhat cumbersome, we introduce a two-place '**B**' predicate that indicates that a world $\mathbf{w}$ "belongs to" an interpretation $\mathbf{I}$, which just means that $\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{W} \in \mathbf{I}$. This conforms to our practice in previous modules of referring to worlds as being "in" interpretations. So we can say that a *DI* interpretation meets the following condition:

$(\Pi \mathbf{w})(\mathbf{BwI} \to (\Sigma \mathbf{w}_i)(\mathbf{Bw_iI} \bigwedge \mathbf{Rww}_i))$.

If a world $\mathbf{w}$ belongs to an interpretation $\mathbf{I}$, then there is an accessible world $\mathbf{w}_i$ (possibly the same as $\mathbf{w}$) in the interpretation as well.

It is easily seen that all *KI*-entailments are *DI*-entailments.

If $\{\gamma_1, \ldots, \gamma_n\} \vDash_{KI} \alpha$, then $\{\gamma_1, \ldots, \gamma_n\} \vDash_{DI} \alpha$.

This is because the class of *DI*-frames is a subset of the class of *KI*-frames: all *DI*-frames are *KI*-frames. So any entailment that holds in all *KI*-frames also holds in all *DI*-frames. In this sense, the semantical system *KI* is *contained in* the semantical system *DI*.

This fact means that the semantical results we have proved for *KI* carry over to *DI* and other systems formed by placing restrictions on the accessibility relation. Specifically, Modal Bivalence, Modal Truth-Functionality, and Closure were proved without in any way taking into account the nature of the accessibility relation in a frame.

The semantical system *DI* is a *stronger* system than *KI*, in that some *DI*-entailments are not *KI*-entailments. In this sense, *DI* is an *extension* of *KI*. Specifically,

$\{\Box\alpha\} \vDash_{DI} \Diamond\alpha$, but $\{\Box\alpha\} \nvDash_{KI} \Diamond\alpha$.

Proof. If $\mathbf{v_I}(\Box\alpha, \mathbf{w})$=$\mathbf{T}$, then for all worlds $\mathbf{w}_i$ accessible to $\mathbf{w}$, $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=$\mathbf{T}$. By the seriality of $\mathbf{R}$, there is a world $\mathbf{w}_i$ such that $\mathbf{Rww}_i$, so there is an accessible world $\mathbf{w}_i$ such that $\mathbf{v_I}(\alpha, \mathbf{w}_i)$=$\mathbf{T}$. So, $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=$\mathbf{T}$.

The non-entailment in *KI* was proved in the last module, and it hinges on the fact that a *K*-frame may contain "dead-end" worlds to which no world is accessible. The requirement that $\mathbf{R}$ be serial prohibits the presence of dead-ends in the set $\mathbf{W}$ of worlds in a frame. There is guaranteed to be a world $\mathbf{w}_i$ accessible to a given world $\mathbf{w}$, which guarantees that $\mathbf{v_I}(\Diamond\alpha, \mathbf{w})$=$\mathbf{T}$. This holds regardless of the choice of $\mathbf{w}$ and $\mathbf{I}$, so long as the frame on which $\mathbf{I}$ is based is serial.

The proof of the entailment can be given using a meta-logical derivation.

Sketch of a semantical proof that: $\{\Box\alpha\} \vDash_{DI} \Diamond\alpha$

1	$\mathbf{v_I}(\Box\alpha, \mathbf{w}) = \mathbf{T}$	Assumption
2	$(\Pi\mathbf{w})(\Sigma\mathbf{w}_i)\mathbf{Rww}_i$	Seriality of $\mathbf{R}$
3	$(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i)$	2 Π E
4	$\quad\mathbf{Rww}_1$	Assumption
5	$\quad(\Pi\mathbf{w}_i)(\mathbf{Rww}_i \rightarrow \mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T})$	1 **SR**-$\Box$
6	$\quad\mathbf{Rww}_1 \rightarrow \mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	5 Π E
7	$\quad\mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	4 6 $\rightarrow$ E
8	$\quad\mathbf{Rww}_1 \bigwedge \mathbf{v_I}(\alpha, \mathbf{w}_1) = \mathbf{T}$	4 7 $\bigwedge$ I
9	$\quad(\Sigma\mathbf{w}_i)(\mathbf{Rww}_i \bigwedge \mathbf{v_I}(\alpha, \mathbf{w}_i) = \mathbf{T})$	8 Σ I
10	$\quad\mathbf{v_I}(\Diamond\alpha, \mathbf{w}) = \mathbf{T})$	9 **SR**-$\Diamond$
11	$\mathbf{v_I}(\Diamond\alpha, \mathbf{w}) = \mathbf{T})$	3 4-10 Σ E

The result can be represented graphically, using two steps. First, we establish that given the truth of $\Box\alpha$ at a world $\mathbf{w}$, α is true at all accessible worlds $\mathbf{w}_i$

$$
\begin{array}{ccc}
\mathbf{w} & \overset{*}{\longrightarrow} & \mathbf{w}_i \\
\Box\alpha & & \\
\hline
\mathbf{T} & & \\
 & & \alpha \\
\hline
 & & \mathbf{T}
\end{array}
$$

Then we consider a world $\mathbf{w}_1$ satisfies the seriality requirement by being accessible to $\mathbf{w}$. We then carry over our result that α is true at all accessible worlds $\mathbf{w}_i$ to world $\mathbf{w}_1$

$$
\begin{array}{ccc}
\mathbf{w} & \overset{\longrightarrow}{*} & \mathbf{w}_1 \\
\Box\alpha & & \\
\hline
\mathbf{T} & & \\
 & & \\
 & & \alpha \\
\hline
 & & \mathbf{T} \\
 & & \\
\Diamond\alpha & & \\
\hline
\mathbf{T} & &
\end{array}
$$

We can now establish another desirable result of the system. There are valid sentences of the form $\Diamond\alpha$. Specifically, if α is valid in *DI*, then so is $\Diamond\alpha$.

If $\vDash_{DI}\alpha$, then $\vDash_{DI}\Diamond\alpha$.

Proof. Suppose $\vDash_{DI}\ \alpha$. Then at all worlds $\mathbf{w}_i$ on all interpretations $\mathbf{I}$ based on any *DI*-frame, $\mathbf{v_I}(\alpha,\ \mathbf{w}_i)$=$\mathbf{T}$. For every world $\mathbf{w}$, there is an accessible world $\mathbf{w}_i$. So, there is an accessible $\mathbf{w}_i$ such that $\mathbf{v_I}(\alpha,\ \mathbf{w}_i)$=$\mathbf{T}$. Hence, $\mathbf{v_I}(\Diamond\alpha,\ \mathbf{w})$=$\mathbf{T}$. Because this result holds for all worlds on all interpretations based on any *DI*-frame, $\vDash_{DI}\Diamond\alpha$, which was to be proved.

2 The Derivational System *DD*

Just as the semantical system *DI* builds on the semantical rules for *KI*, the derivational system *DD* adds a rule to the derivational rules for *KD*, with the aim of allowing the derivation of $\Diamond\alpha$ from $\{\Box\alpha\}$. As before, the derivational rule will closely follow the semantical rule. In the semantical system, if a necessity-sentence $\Box\alpha$ is given the value **T** at world **w**, not only is α true at any arbitrary accessible world if there is one, but it is true at at least one accessible world. So when we write down a restricted scope line indicating an accessible world and get a result α there, we can end the restricted scope line and write down $\Diamond\alpha$. We shall call this a "weak" rule of $\Diamond$ Introduction, because we will later introduce a stronger rule.

Weak $\Diamond$ Introduction

$$\begin{array}{|l|l} \Box & \vdots \\ & \alpha \end{array}$$

$$\Diamond\alpha \qquad \text{W}\Diamond\ \text{I}$$

Provided that α is not in the scope of any assumption within the strict scope line.

We assert without proof that the derivational system *DD* resulting from adding this rule to the derivational rules of *KD* is complete with respect to the semantical system *DI*. We also assert without proof that the derivational system is sound. This claim can be motivated by the way in which a derivation mirrors the semantical reasoning used in the meta-logical derivation above.

To prove: $\Box\alpha \vdash_{DD} \Diamond\alpha$

1	$\Box\alpha$	Assumption
2	$\Box \mid\ \alpha$	1 SR-$\Box$
3	$\Diamond\alpha$	2 W $\Diamond$ I

Some texts give as rule the derivation of $\Diamond\alpha$ directly from $\Box\alpha$. This can be treated in *DD* as a derived rule, as is clear from the preceding derivation.

If the alternative rule is taken as primitive, then our rule of Weak $\Diamond$ Introduction would be a derived rule. It is easily seen why it would be.

Weak $\Diamond$ Introduction as a derived rule

$$\begin{array}{|l|l} \Box & \vdots \\ & \alpha \end{array}$$

$$\begin{array}{ll} \Box\alpha & \Box\ \text{I} \\ \Diamond\alpha & \text{Alternative Primitive Rule} \end{array}$$

If Weak $\Diamond$ Introduction is adopted as a primitive rule, then system *D* requires a primitive possibility operator in its syntax. If only the '$\Box$' operator is taken as primitive (and sentences with '$\Diamond$' as their main operator are defined in terms of sentences with the '$\Box$' operator), a new '$\Box$' rule is needed. The most natural rule would be modeled on the Impossibility Introduction rule. That is, if $\sim\alpha$ occurs inside a single restricted scope line, and not in the scope of any undischarged assumptions, then the scope line may be ended and '$\sim\Box\alpha$' written.

$\sim\Box$ Introduction

$$\begin{array}{l|l}
\Box & \vdots \\
& \sim\alpha \\
\hline
\multicolumn{2}{l}{\sim\Box\alpha \qquad \sim\Box\ \mathrm{I}}
\end{array}$$

Provided that $\sim\alpha$ is not in the scope of any assumption within the restricted scope line.

Exercise. Give a justification for this rule using the semantical rules for *DI*.

Either rule can be derived from the other given Duality as a replacement rule. Here is how the $\sim\Box$ Introduction rule can be derived using the Weak $\Diamond$ Introduction rule. Assume that $\sim\alpha$ occurs within a restricted scope line. Then the scope line can terminated in favor of $\Diamond\sim\alpha$. With Duality, this is shown to be equivalent to $\sim\Box\alpha$.

$\sim\Box$ Introduction as a derived rule

$$\begin{array}{l|l}
\Box & \vdots \\
& \sim\alpha \\
\hline
\multicolumn{2}{l}{\Diamond\sim\alpha \qquad \mathrm{W}\ \Diamond\ \mathrm{I}} \\
\multicolumn{2}{l}{\sim\Box\alpha \qquad \text{Duality}}
\end{array}$$

Exercise. Show that with Duality as a derived rule and $\sim\Box$ Introduction, the rule of Weak $\Diamond$ Introduction can be derived.

It might be noticed that neither Weak $\Diamond$ Introduction nor $\sim\Box$ Introduction requires that any sentence of the form $\Box\alpha$ occur outside the restricted scope line. In this way, the two rules deviate from the semantical proof that $\{\Box\alpha\} \vDash_D \Diamond\alpha$. This does not threaten to make the rule unsound, however. If a sentence of the form $\Box\alpha$ does occur outside the restricted scope line, then the semantical reasoning is followed perfectly. If it does not, there are no untoward consequences.

If we can derive α entirely with the strict scope line, then $\vdash_{DD}\alpha$. More generally, It is easy to see that in general, if $\vdash_D \alpha$ then $\vdash_D \Diamond\alpha$, which parallels the result from the semantical system. Here is an example.

To Prove: $\vdash_{DD}\Diamond\sim(P \wedge \sim P)$

1	$\Box$ | | $P \wedge \sim P$	Assumption
2	| | P	1 $\wedge$ E
3	| | $\sim P$	1 $\wedge$ E
4	| $\sim(P \wedge \sim P)$	1-3 $\sim$ I
5	$\Diamond\sim(P \wedge \sim P)$	1-4 W $\Diamond$ I

Now the last step of this derivation could just as well have been, by $\Box$ Introduction, $\Box(\sim(P \wedge \sim P)$. So we could have derived this result first, which would then have allowed $\Box\alpha$ to occur outside the restricted scope line. Then the semantical reasoning used above would be reflected in the derivation of $\Diamond\sim(P \wedge \sim P)$.

3 The Axiom System *D*

The axiom system *D* is obtained by adding to the axioms of *K* the further axiom schema:

$\vdash_D \Box\alpha \supset \Diamond\alpha.$

This axiom is clearly valid in *DI*, by the extending the reasoning that showed that $\{\Box\alpha\} \vDash_{DI} \Diamond\alpha$. Assume that $\Box\alpha$ is true at an arbitrary world **w** in an arbitrary interpretation **I** based on a *DI*-frame. Then α is true at all accessible worlds **w***I*. There is such an accessible world, by the seriality of **R**, so α is true at that world, which makes $\Diamond\alpha$ true at **w**. Thus by **SR**-$\supset$, $\Box\alpha \supset \Diamond\alpha$ is true at **w**, which was to be proved.

4 Applications of the *D*-Systems

In the last chapter, we considered the use of the *K*-systems to represent various modalities of which we have informal conceptions. In every case, it appeared that a stronger system is needed. We shall now look at the adequacy of the *D*-systems to represent the various modalities treated in this text.

The *D*-systems yield a result which is amenable to the notion of logical necessity. What is true of logical necessity (by virtue of "the laws of logic" or by its "logical form") should at least be possibly true. With respect to hypothetical necessity, the *D*-systems do not allow worlds at which everything is trivially necessary because nothing is impossible. If we take the accessibility relation as specifying a condition that might or might not hold relative to a world, we must then say that the condition can be satisfied, in that it is satisfied at an accessible world. It seems plausible on the face of it that given almost any condition, this would be a desirable result.

There is a change in the *D*-systems which is of some small significance for the strict-implication interpretation of the fish-hook. In the semantical system *KI*, all sentences of the form $\alpha \multimap \beta$ are true at dead-ends. This result can be obtained in two ways. First, $\Box\beta$ is always true at a dead-end, and if $\Box\beta$ is true at a world, then $\alpha \multimap \beta$ is also true of that world. Second, $\sim\Diamond\alpha$ is true at a dead end, and any strict-implication sentence with an impossible antecedent at a world is true at that world. Ridding the semantics of dead-ends does away with this peculiar way of generating necessities and impossibilites. But the "paradoxes of strict implication" remain in the *D*-systems, and indeed in any systems based on the *K*-systems. If, for more orthodox reasons, $\Box\alpha$ is true at a world, then so is $\beta \multimap \alpha$, and so forth.

The deontic interpretation seems to require the restriction on accessibility laid down by semantical system *D*. If something is obligatory relative to a world, it should be treated as being permissible relative to that world as well.[2] We have this result because $\{\mathrm{O}\alpha\} \vDash_{DI} \mathrm{P}\alpha$. This is about as strong as we want a system of deontic logic to be. Indeed, the name of the axiom system, '*D*,' is an abbreviation for 'deontic.'

It should be noted that it is possible meaningfully to combine the deontic modal operators with other modal operators, which would produce a system with more expressive power. Thus, we may wish to assert that what is obligatory is possible (or that "ought implies can"), so that $\mathrm{O}\alpha$ should entail $\Diamond\alpha$. Another option is to combine the denotic operators with temporal operators. We shall not explore these combined modalities in this text.

As was stated in the last module, it seems desirable that if a subject believes that α at a time, then α is compatible with what the subject believes. We have this result in *DI*: $\{\mathrm{B}_{s,t}\alpha\} \vDash_{DI} \mathrm{P}_{s,t}\alpha$. Moreover, there are no *DI*-frames containing dead-ends, so if $\mathrm{B}_{s,t}\alpha$ is true at a world, it is because there are accessible worlds at which α holds, and it holds at all of them. This is a much more realistic view of belief than one that yields a belief when nothing is compatible with what one believes. The same considerations hold for knowledge, insofar as it requires belief.

[2] Permissibility, which may be defined in terms of obligation (or *vice-versa)*, must therefore be understood in the same way in the semantics. Some would claim, for example, that there are acts which are legally obligatory but not morally permissible.

It is also obvious that a temporal logic of the future and/or the past can only be adequately represented in systems at least as strong as the *D*-systems. It is plausible to say that if it always will be the case that α, then it will at some time be the case that α. If there is a future at all, then it contains some times. Less obvious is the requirement in the semantics that for each time, there is a future time, which is what would be represented by a serial accessibility relation where an accessible world is a future world. Note that if the number of times is finite, the serial relation would lead to some kind of loop, where an accessible world is either present or past. In that case, either the present or the past lies in the future. If there are infinitely many times, then the seriality condition requires that there be a future time for any time, which is as it should be. The same remarks hold, *mutatis mutandis*, for what has at all times been the case.