Module 9
S4 and Equivalent Systems

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Contents

Thus far, we have examined families of equivalent systems of increasing strength, where a stronger system in the family contains all the weaker ones. The axiom system *S4* is an extension of the system *T* and thus contains *T* as well as the axiomatic systems *D* and *K*.[1]

The semantical system *S4I* is obtained by adding to the reflexivity condition of *TI* a second condition on accessibility, transitivity. Transitivity does not imply either reflexivity or seriality. So the transitivity condition could just as well be added to the semantical system *DI* to yield a system without reflexivity. Or it could be added to *KI*, yielding a still-weaker system with transitivity as the only condition on accessibility. We shall here consider the classical *S4*-systems built on the *T*-systems.

1 The Semantical System *S4I*

As just noted, the semantical system *S4I* is just like the system *TI* except for the requirement that **R** be *transitive* as well as being reflexive. The transitive character of a relation **R** is defined as follows:

R is **transitive** iff $(\Pi \mathbf{x})(\Pi \mathbf{y})(\Pi \mathbf{z})((\mathbf{Rxy} \wedge \mathbf{Ryz}) \rightarrow \mathbf{Rxz})$.

Applied to frames, this means that if a world $\mathbf{w}_i$ is accessible to $\mathbf{w}$, and $\mathbf{w}_j$ is accessible to $\mathbf{w}_i$, then $\mathbf{w}_j$ is accessible to $\mathbf{w}$.

We can define an *S4I*-frame as a set $\langle \mathbf{W}, \mathbf{R} \rangle$, such that :

[1] *S4* is also an extension of Lewis's non-normal system *S3*.

$(\Pi \mathbf{w})(\mathbf{w} \in \mathbf{W} \rightarrow \mathbf{Rww})$, and

$(\Pi \mathbf{w})(\Pi \mathbf{w}_i)\Pi \mathbf{w}_j)((((\mathbf{BwI} \bigwedge \mathbf{Bw}_i\mathbf{I}) \bigwedge \mathbf{Bw}_j\mathbf{I}) \bigwedge(\mathbf{Rww}_i \bigwedge \mathbf{Rw}_i\mathbf{w}_j)) \rightarrow \mathbf{Rww}_j)$.

The transitivity of the accessibility relation in *S4I*-frames yields one of the characteristic consequence relations of the *S4*-systems.

$\{\Box\alpha\} \vDash_{S4I} \Box\Box\alpha$.

The proof of the entailment can be given using a meta-logical derivation.

Sketch of a semantical proof that: $\{\Box\alpha\} \vDash_{S4I} \Box\Box\alpha$

1	$\mathbf{v_I}(\Box\alpha, \mathbf{w}) = \mathbf{T}$	Assumption
2	$(\mathbf{Rww}_1 \bigwedge \mathbf{Rw}_1\mathbf{w}_2) \rightarrow \mathbf{Rww}_2$	Transitivity of **R**
3	$\quad \mathbf{Rww}_1$	Assumption
4	$\quad\quad \mathbf{Rw}_1\mathbf{w}_2$	Assumption
5	$\quad\quad \mathbf{Rww}_1 \bigwedge \mathbf{Rw}_1\mathbf{w}_2$	3 4 $\bigwedge$ I
6	$\quad\quad \mathbf{Rww}_2$	2 5 $\rightarrow$ E
7	$\quad\quad \mathbf{Rww}_2 \bigwedge \mathbf{v_I}(\Box\alpha, \mathbf{w}) = \mathbf{T}$	1 6 $\bigwedge$ I
8	$\quad\quad \mathbf{v_I}(\alpha, \mathbf{w}_2) = \mathbf{T}$	7 **SR**-$\Box$
9	$\quad \mathbf{Rw}_1\mathbf{w}_2 \rightarrow \mathbf{v_I}(\alpha, \mathbf{w}_2) = \mathbf{T}$	4-8 $\rightarrow$ I
10	$\quad \mathbf{v_I}(\Box\alpha, \mathbf{w}_1) = \mathbf{T}$	9 **SR**-$\Box$
11	$\mathbf{Rww}_1 \rightarrow \mathbf{v_I}(\Box\alpha, \mathbf{w}_1) = \mathbf{T}$	3-10 $\rightarrow$ I
12	$\mathbf{v_I}(\Box\Box\alpha, \mathbf{w}) = \mathbf{T}$	11 **SR**-$\Box$

The transitivity of accessibility can be depicted graphically by placing an arrow above the name of the "connecting" world, indicating that the third world is accessible to this first.

$$w_1 \longrightarrow \overset{\longrightarrow}{w_2} \longrightarrow w_3$$

This will allow us to use a modal truth-table to demonstrate the entailment proved above. We will assume that an arbitrary world $\mathbf{w_1}$ is accessible to $\mathbf{w}$ and that an arbitrary world $\mathbf{w_2}$ is accessible to $\mathbf{w_1}$. Then since these worlds are arbitrary, the relation that holds by transitivity between $\mathbf{w}$ and $\mathbf{w_2}$ is perfectly general and thus merits a '*' above it.

$$\begin{array}{ccccc}
\mathbf{w} & \overset{*}{\longrightarrow} & \overset{\overset{*}{\longrightarrow}}{\mathbf{w}_1} & \overset{*}{\longrightarrow} & \mathbf{w}_2 \\
\Box\alpha & & & & \\
\hline
\mathbf{T} & & & & \\
 & & & & \alpha \\
\hline
 & & & & \mathbf{T} \\
 & & \Box\alpha & & \\
\hline
 & & \mathbf{T} & & \\
\Box\Box\alpha & & & & \\
\hline
\mathbf{T} & & & &
\end{array}$$

The second characteristic consequence relation in *S4I* uses the possibility operator:

$\{\Diamond\Diamond\alpha\} \vDash_{S4I} \Diamond\alpha$.

A modal truth-table will illustrate why this holds.

$$
\begin{array}{ccccc}
 & & \overset{\longrightarrow}{*} & & \\
\mathbf{w} & \overset{\longrightarrow}{*} & \mathbf{w}_1 & \overset{\longrightarrow}{*} & \mathbf{w}_2 \\
\Diamond\Diamond\alpha & & & & \\
\hline
\mathbf{T} & & & & \\
 & & \Diamond\alpha & & \\
\hline
 & & \mathbf{T} & & \\
 & & & & \alpha \\
\hline
 & & & & \mathbf{T} \\
\Diamond\alpha & & & & \\
\hline
\mathbf{T} & & & &
\end{array}
$$

Since the class of *S4I*-frames is a subset of the class of *TI*-frames, any entailment that holds in all *TI*-frames also holds in all *S4I*-frames. So all *TI*-entailments (and hence all *DI*- and *KI*-entailments) are *S4I*-entailments.

If $\{\gamma_1, \ldots, \gamma_n\} \vDash_{TI} \alpha$, then $\{\gamma_1, \ldots, \gamma_n\} \vDash_{S4I} \alpha$.

Thus, the semantical system *TI* is contained in the semantical system *S4I*.

S4I is a stronger system than *TI*, in that some *S4I*-entailments are not *TI*-entailments, because some *TI*-frames are not *S4I*-frames. So *S4I* is an extension of *TI* (and hence of *DI* and *KI*). Specifically,

$\{\Box\alpha\} \nvDash_{TI} \Box\Box\alpha$.

Proof. Let W in a frame Fr contain three worlds, w_1, w_2, and w_3, such that Rw_1w_1, Rw_2w_2, Rw_3w_3, Rw_1w_2, and Rw_2w_3. R is therefore reflexive, and so Fr is a *TI*-frame. Now let $\mathbf{v_I}(\alpha, w_1) = \mathbf{T}$, $\mathbf{v_I}(\alpha, w_2) = \mathbf{T}$, and $\mathbf{v_I}(\alpha, w_3) = \mathbf{F}$. If follows from **SR**-$\Box$ that $\mathbf{v_I}(\Box\alpha, w_1) = \mathbf{T}$, since α is true at all worlds accessible to w_1, i.e., at w_1 itself and at w_2. Since $\mathbf{v_I}(\alpha, w_3) = \mathbf{F}$, $\mathbf{v_I}(\Box\alpha, w_2) = \mathbf{F}$, and hence $\mathbf{v_I}(\Box\Box\alpha, w_1) = \mathbf{F}$, by **SR**-$\Box$.

$$
\begin{array}{ccccc}
\curvearrowleft & & \curvearrowleft & & \curvearrowleft \\
w_1 & \overset{*}{\longrightarrow} & w_2 & \overset{*}{\longrightarrow} & w_3 \\
\Box\alpha & & & & \\
\hline
\mathbf{T} & & & & \\
 & & \alpha & & \\
\hline
 & & \mathbf{T} & & \\
 & & & & \alpha \\
\hline
 & & & & \mathbf{F} \\
 & & \Box\alpha & & \\
\hline
 & & \mathbf{F} & & \\
\Box\Box\alpha & & & & \\
\hline
\mathbf{F} & & & &
\end{array}
$$

The semantical system *S4I* builds on some consequences in *TI* for iterated modalities. For the '$\Box$' operator, we proved that:

$\{\Box\Box\alpha\} \vDash_{TI} \Box\alpha$, and so,

$\{\Box\Box\alpha\} \vDash_{S4I} \Box\alpha$.

We have just proved that

$\{\Box\alpha\} \vDash_{S4I} \Box\Box\alpha$.

Putting the two results together, we can say that any sentence which is preceeded by a string of two '□' symbols is equivalent to a sentence containing only one, and *vice-versa*. This result may be generalized to sentences beginning with strings of more than two '□' operators: each is equivalent to a sentence beginning with one less operator in the string, and hence ultimately to a sentence beginning with only one '□' operator.

A corresponding result holds for the '◊' symbol. We saw that in *TI* and hence in *S4I*:

$\{\Diamond\alpha\} \vDash_{S4I} \Diamond\Diamond\alpha$.

We have just illusrated a proof that:

$\{\Diamond\Diamond\alpha\} \vDash_{S4I} \Diamond\alpha$.

So we have the same reduction as for the '□.' Any sentence preceeded by a string of at least two '◊' symbols is equivalent to a sentence containing one fewer '◊'. The general reduction of any number of '□ operators at the beginning of a sentence to only one applies to the '◊' operator as well.

It must be noted, however, that this result cannot be generalized any further. If we allowed a sentence preceded by just one modal operator to be equivalent to one containing no modal operator, *S4I* would be equivalent to *SI*.

There are two further "reductions" that hold in *S4I*. A sentence of the form $\Diamond\Box\Diamond\Box\alpha$ is equivalent to a sentence of the form $\Diamond\Box\alpha$, and a sentence of the form $\Box\Diamond\Box\Diamond\alpha$ is equivalent to one with the form $\Box\Diamond\alpha$.[2] Sentences of the form $\Diamond\Box\Diamond\alpha$ and $\Box\Diamond\Box\alpha$ are not reducible to any equivalents with fewer modal operators, nor are sentences of the form $\Box\Diamond\alpha$ and $\Diamond\Box\alpha$. This means that in *S4* there are six non-equivalent modal sentence forms: $\Box\alpha$, $\Diamond\alpha$, $\Box\Diamond\alpha$, $\Diamond\Box\alpha$, $\Box\Diamond\Box\alpha$, and $\Diamond\Box\Diamond\alpha$.

Exercise. Give counter-examples to prove that the four forms just claimed not to be reducible are not reducible.

2 The Derivational System *S4D*

Just as the semantical system for *S4I* contains the semantical rules inherited from *KI*, *DI*, and *TI*, the derivational system inherits the derivational rules from these systems. An additional rule generates the effects of transitivity of accessibility. It will be a special version of Strict Reiteration for the '□,' SR-□(4), which allows that when $\Box\alpha$ occurs, α may be reiterated across *two* restricted scope lines.

Strict Reiteration for '□' (4)

$$
\begin{array}{|l l}
\quad\Box\alpha & \text{Already derived} \\
\Box\begin{array}{|l} \vdots \\ \Box\begin{array}{|l} \vdots \\ \\ \alpha \end{array} \end{array} & \\
 & \text{SR-}\Box(4)
\end{array}
$$

This rule simulates the result at line 8 of the meta-logical derivation of $\{\Box\alpha\} \vDash_{S4I} \Box\Box\alpha$. The left restricted scope line represents the assumption that $\mathbf{Rww}_1$ and the nested one the further assumption that $\mathbf{Rw}_1\mathbf{w}_2$. The fact that α may be strictly reiterated across the two scope lines reflects transitivity.[3]

[2] This will be proved in the section on the derivational system *S4D*.

[3] An equivalent rule would allow the strict reiteration across any number of scope lines. The current rule is adopted in order to reflect the definition of transitivity.

We assert without proof that the derivational system *S4D* resulting from adding this rule to the derivational rules of *S4D* is complete with respect to the semantical system *S4I*. We also assert without proof that the derivational system is sound. This claim can be motivated by the way in which a derivation mirrors the semantical reasoning used in the meta-logical derivation above.

To prove: $\square\alpha \vdash_{S4D} \square\square\alpha$

1	$\mid \square\alpha$	Assumption
2	$\mid \square\mid \square\mid\ \alpha$	1 SR-$\square$(4)
3	$\mid\ \mid\ \square\alpha$	2 $\square$ I
4	$\mid \square\square\alpha$	2-3 $\square$ I

Fitch himself used a different rule of Strict Reiteration for his *S4* derivational system.

Strict Reiteration for '$\square$' (4), Fitch

$\mid \square\alpha$	Already derived
$\mid \square\mid\ \vdots$	
$\mid\ \mid\ \square\alpha$	SR-$\square$(4)F

This is a derived rule in the present system, as should be obvious from the previous derivation. It also can be seen from the following derivation-schema.

Fitch's Strict Reiteration for '$\square$' (4) as a derived rule

$\mid \square\alpha$	Already derived
$\mid \square\mid\ \vdots$	
$\mid\ \mid \square\mid\ \vdots$	
$\mid\ \mid\ \mid\ \alpha$	SR-$\square$(4)
$\mid\ \mid\ \square\alpha$	$\square$ I

The rule given in this text clearly brings out the role of accessibility and the way it works with the transitivity relation imposed by *S4I*-frames. Fitch's rule suppresses the role of accessibility in determining truth of a modal sentence at a world, and perhaps it is for this reason that Fitch himself did not see how his strict scope lines were a clue to modern possible-worlds semantics.

An alternative rule, for systems with '$\Diamond$' as primitive, could be a rule of strict reiteration for '$\sim\Diamond$. Such a rule is obviously equivalent to SR-$\square$(4) given Duality.

Strict Reiteration for '$\sim\Diamond$' (S4)

$\mid \sim\Diamond\alpha$	Already derived
$\mid \square\mid\ \vdots$	
$\mid\ \mid \square\mid\ \vdots$	
$\mid\ \mid\ \mid\ \sim\alpha$	SR-$\sim\Diamond$(4)

Using this rule we can prove the '$\Diamond$' counterpart of the characteristic consequence of *S4D*: $\{\Diamond\Diamond\alpha\} \vdash_{A4D} \Diamond\alpha$.

To prove: $\{\Diamond\Diamond\alpha\} \vdash_{S4D} \Diamond\alpha$

1	$\mid \Diamond\Diamond\alpha$	Assumption
2	$\mid\ \mid {\sim}\Diamond\alpha$	Assumption
3	$\mid\ \mid \Box\mid \Box\mid\ {\sim}\alpha$	2 SR-~◇(4)
4	$\mid\ \mid\ \ \mid {\sim}\Diamond\alpha$	3 ~◇ I
5	$\mid\ \mid {\sim}\Diamond\Diamond\alpha$	3-4 ~◇ I
6	$\mid\ \mid \Diamond\Diamond\alpha$	1 Reiteration
7	$\mid \Diamond\alpha$	2-6 ~ E

For the unnegated possibility operator, we will change the rule of $\Diamond$ Elimination. The effect of transitivity is that when γ is true at a world $\mathbf{w}_j$ accessible to a world $\mathbf{w}_i$ accessible to $\mathbf{w}$, then $\Diamond\gamma$ is true at $\mathbf{w}$. As a result, $\Diamond\gamma$ is true at $\mathbf{w}_j$. , $\Diamond\gamma$ is true at $\mathbf{w}$. The appeareance of the sentence γ in the last step (not in the scope of any other assumptions) in the scope of a strict assumption in the scope of a further strict assumption allows us to terminate inner strict assumption line and write down γ itself outside it. We require in addition that strict assumptions be made, which in turn requires that a sentence $\Diamond\alpha$ occur outside the first restricted scope line and some formula $\Diamond\beta$ occur outside the inner restricted scope line, and that both are "instantiated" at their respective scope lines.

$\Diamond$ Elimination (4)

$\mid \Diamond\alpha$	
$\mid \Box\mid \alpha$	
$\mid\ \ \mid\ \vdots$	
$\mid\ \ \mid \Diamond\beta$	
$\mid\ \ \mid\ \vdots$	
$\mid\ \ \mid \Box\mid \beta$	
$\mid\ \ \mid\ \ \mid\ \vdots$	
$\mid\ \ \mid\ \ \mid \gamma$	
$\mid \Diamond\gamma$	$\Diamond$ E (4)

Provided that γ is not in the scope of any assumption within the restricted scope line.

The rule is symmetrical with the Strict Reiteration rule for the '$\Box$,' which allows a jump across two restricted scope lines at once, in that it ends two scope lines simultaneously.

Exercise. Prove that this rule is a derived rule given SR-$\Box$(4) and Duality.

Note that this rule does not prohibit crossing restricted scope lines in which an assumption has not been made.

Exercise. Use the semantical system *S4I* to show why no such prohibition is needed.

Using this rule we can prove that $\{\Diamond\Diamond\alpha\} \vdash_{S4D} \Diamond\alpha$.

To prove: $\{\Diamond\Diamond\alpha\} \vdash_{S4D} \Diamond\alpha$

1	$\mid\ \Diamond\Diamond\alpha$	Assumption
2	$\mid\ \Box\mid\ \Diamond\alpha$	1 SR-$\Diamond$
3	$\mid\ \mid\ \Box\mid\ \alpha$	2 SR-$\Diamond$
4	$\mid\ \mid\ \mid\ \alpha$	3 Reiteration
5	$\mid\ \Diamond\alpha$	1 2-4 $\Diamond$ E

With our full complement of rules in hand, we can also derive some other consequence relations that were announced in the semantical section above.

To prove: $\{\Box\Diamond\Box\Diamond\alpha\} \vdash_{S4D} \Box\Diamond\alpha$

1	$\mid\ \Box\Diamond\Box\Diamond\alpha$	Assumption
2	$\mid\ \Box\mid\ \Diamond\Box\Diamond\alpha$	1 SR-$\Box$
3	$\mid\ \mid\ \Box\mid\ \Box\Diamond\alpha$	2 SR-$\Diamond$
4	$\mid\ \mid\ \mid\ \Diamond\alpha$	3 $\Box$ E
5	$\mid\ \mid\ \mid\ \Box\mid\ \alpha$	4 SR-$\Diamond$
6	$\mid\ \mid\ \mid\ \mid\ \alpha$	5 Reiteration
7	$\mid\ \mid\ \Diamond\alpha$	2 3-6 $\Diamond$ E
8	$\mid\ \Box\Diamond\alpha$	2-7 $\Box$ I

To prove: $\{\Box\Diamond\alpha\} \vdash_{S4D} \Box\Diamond\Box\Diamond\alpha$

1	$\mid\ \Box\Diamond\alpha$	Assumption
2	$\mid\ \Box\mid\ \Diamond\alpha$	1 SR-$\Box$
3	$\mid\ \mid\ \Box\mid\ \Diamond\alpha$	1 SR-$\Box$(4)
4	$\mid\ \mid\ \Box\Diamond\alpha$	3 $\Box$ I
5	$\mid\ \mid\ \Diamond\Box\Diamond\alpha$	4 S $\Diamond$ I
6	$\mid\ \Box\Diamond\Box\Diamond\alpha$	3-5 $\Box$ I

To prove: $\{\Diamond\square\Diamond\square\alpha\} \vdash_{S4D} \Diamond\square\alpha$

1	$\mid\ \Diamond\square\Diamond\square\alpha$	Assumption
2	$\mid\ {}^{\square}\mid\ \square\Diamond\square\alpha$	1 SR-$\Diamond$
3	$\mid\ \ \mid\ \Diamond\square\alpha$	2 $\square$ E
4	$\mid\ \ \mid\ {}^{\square}\mid\ \square\alpha$	3 SR-$\Diamond$
5	$\mid\ \ \mid\ \ \mid\ \square\alpha$	4 Reiteration
6	$\mid\ \Diamond\square\alpha$	1 2-5 $\Diamond$ E(4)

To prove: $\{\Diamond\square\alpha\} \vdash_{S4D} \Diamond\square\Diamond\square\alpha$

1	$\mid\ \Diamond\square\alpha$	Assumption
2	$\mid\ {}^{\square}\mid\ \square\alpha$	1 SR-$\Diamond$
3	$\mid\ \ \mid\ {}^{\square}\mid\ {}^{\square}\mid\ \alpha$	2 SR-$\square$(4)
4	$\mid\ \ \mid\ \ \mid\ \square\alpha$	4 $\square$ I
5	$\mid\ \ \mid\ \ \mid\ \Diamond\square\alpha$	5 S $\Diamond$ I
6	$\mid\ \ \mid\ \square\Diamond\square\alpha$	3-6 $\square$ I
7	$\mid\ \Diamond\square\Diamond\square\alpha$	2 3-7 $\Diamond$ E

Finally, we may state a rule for the '$\prec$.'

Strict Reiteration for '$\prec$' (S4)

$\mid\ \alpha \prec \beta$	Already derived
$\mid\ {}^{\square}\mid\ \vdots$	
$\mid\ \ \mid\ {}^{\square}\mid\ \vdots$	
$\mid\ \ \mid\ \ \mid\ \alpha \supset \beta$	SR-$\prec$(4)

This rule is similar to the '$\prec$' strict reiteration rule for *KD*. The difference is that in *S4D*, one is allowed to reiterate across two restricted scope lines, just as with the rule Strict Reiteration-$\square$(4).

Here is an a derivation of a result that will be discussed in the section on applications.

To prove: $\vdash_{S4D} (\alpha \strictif \beta) \strictif ((\beta \strictif \gamma) \strictif (\alpha \strictif \gamma))$

1	$\Box\vert\ \alpha \strictif \beta$	Assumption
2	$\vert\ \Box\vert\ \beta \strictif \gamma$	Assumption
3	$\vert\ \vert\ \Box\vert\ \alpha$	Assumption
4	$\vert\ \vert\ \vert\ \alpha \supset \beta$	1 SR-$\strictif$(4)
5	$\vert\ \vert\ \vert\ \beta \supset \gamma$	2 SR-$\strictif$
6	$\vert\ \vert\ \vert\ \beta$	3 4 $\supset$ E
7	$\vert\ \vert\ \vert\ \gamma$	5 6 $\supset$ E
8	$\vert\ \vert\ \alpha \strictif \gamma$	3-7 $\strictif$ I
9	$\vert\ (\beta \strictif \gamma) \strictif (\alpha \strictif \gamma)$	2-8 $\strictif$ I
10	$(\alpha \strictif \beta) \strictif ((\beta \strictif \gamma) \strictif (\alpha \strictif \gamma))$	1-9 $\strictif$ I

3 The Axiom System *S4*

The axiom system *S4* is obtained by adding to the axiom schemata of *T* the further axiom schema:

$\vdash_{S4} \Box\alpha \supset \Box\Box\alpha.$

This axiom is clearly valid in *S4I*, by reasoning similar to that used in showing the entailment $\{\Box\alpha\} \vDash_{S4I} \Box\Box\alpha$. Alternatively, if the '$\Diamond$' operator is primitive, the axiom schema would be:

$\vdash_{S4} \Diamond\alpha \supset \Diamond\Diamond\alpha.$

4 Applications of the *S4*-Systems

4.1 Alethic Modal Logic

In the *S4*-systems, the logical modalities of possibility and necessity are "reduced" in the sense that a sentence whose main logical operator is a '$\Diamond$' or a '$\Box$' is equivalent to a sentence to which any number of the same kind of operators is prefixed. Given our informal interpretation of *MSL* necessity-sentences, the reduction implies that it is true by the laws of logic that α if and only if it is true by the laws of logic that it is true by the laws of logic that α, etc.

This seems desirable for a system of logical possibilities and necessities just in case the "laws of logic" that govern modalities are the same as the "laws of logic" that govern non-modal sentences. So if α is non-modal, and is true by the "laws of logic," then the fact that it is true by the "laws of logic" is itself a consequence of the same "laws of logic." So the question of suitability comes down to whether one accepts a monolithic or a hierarchical set of "laws of logic."

For hypothetical necessity, whether to impose a restriction on accessibility depends on one's purposes. Put abstractly, it may be the case that when we impose a condition on a set of worlds, there may be something about those worlds such that worlds accessible to them satisfy a different condition.

Hughes and Cresswell consider the condition of conceivability that can be represented by the accessibility relation.[4] We can think of conceivability as hypothetical possibility. If some state of affairs α is

[4] *An Introduction to Modal Logic*, pp. 77-79.

conceivable at **w**, α holds at some world accessible to **w**. However, it may be that the (conceivable) inhabitants of such an accessible world can conceive of something that cannot be conceived of at the original world. Such a situation could not be represented in *S4I*, given the transitivity of accessibility.

With respect to our running example of laws of nature, it may be that a world is "possible" with respect to **w** in the sense that all the laws of nature applicable in **w** hold in it. This might mean that any phenomena at the accessible worlds that are appropriate for those laws of w fall under them. But there may be other phenomena that do not fall under those laws, and there may be other laws, not applicable at **w**, which govern them at various accessible worlds. Then there may be worlds accessible to them that are governed by those laws, which are not part of the set of laws at **w**.

4.2 Conditional Logic

The strict implication operator in axiom system *S4* was thought by Lewis to be too strong to represent logical implication. The problem he raised pertains to a theorem of his system *S3*, but the theorem in question also holds in *S4* (and in *S4D*, as has been shown):

$\vdash_{S4} (\alpha \multimap \beta) \multimap ((\beta \multimap \gamma) \multimap (\alpha \multimap \gamma))$.

This is very similar to a theorem of *T* (and also Lewis's favored system *S2*) that Lewis found acceptable:

$\vdash_T ((\alpha \multimap \beta) \wedge (\beta \multimap \gamma)) \multimap (\alpha \multimap \gamma)$.[5]

The relation between the two might be seen more easily if we put them in terms of the derivability relation:

$\{\alpha \multimap \beta\} \vdash_{S4D} (\beta \multimap \gamma) \multimap (\alpha \multimap \gamma)$, and:

$\{\alpha \multimap \beta, \beta \multimap \gamma\} \vdash_{TD} \alpha \multimap \gamma$.

The derivability relation in *TD* can be understood as a form of inference, "hypothetical syllogism," that has been generally accepted since it was first enunciated by the ancient Stoics. The disputed derivability relation in *S4D* is not so easy to read. We begin with a premise about an implication and conclude from this that one implication implies another. As Lewis put it, "it gives the inference $(q \multimap r) \multimap (p \multimap r)$ whenever $(p \multimap q)$ is a premise" (*Symbolic Logic*, Appendix Two, p. 496). This inference is "dubious," Lewis maintained, unless it is just another way of stating hypothetical syllogism.

What seems to have been bothering Lewis was the fact that the conclusion of the inference is about an implicational relation holding between two sentences which themselves are implication-sentences. To be sure, such an inference seems somewhat unnatural. Suppose I take as a premise the fact that the sentence '$P \wedge Q$' follows from 'P.' I would not have much occasion to infer as a conclusion: it follows from the premise that 'R' follows from 'P' that 'R' follows from '$P \wedge Q$.' But though this seems quite forced, there is nothing obviously wrong with it as an inference. W. T. Parry has noted that even in *S2* (and also in *T*):

If $\vdash_T \alpha \multimap \beta$, then $\vdash_T (\beta \multimap \gamma) \multimap (\alpha \multimap \gamma)$.

So in Lewis's own favored system, one should be entitled to make the "dubious" inference from a premise that is a theorem to the theoremhood of the conclusion.

If there is a problem here, it seems to begin with exportation.[6] Even system *T* does not allow that $\alpha \multimap (\beta \multimap \gamma)$ follows from $(\alpha \wedge \beta) \multimap \gamma$. If this consequence were allowed, it would generate a result that Lewis regarded as unacceptable. From the true sentence '$(P \wedge Q) \multimap P$' we could derive '$P \multimap (Q \multimap P)$'. In that case, strict implication is no less paradoxical than "material implication," which understands the '$\supset$' operator as implication. '$P \supset (Q \supset P)$' is valid in *SL* and consequently in the modal systems based on *SL*.

[5] An inference from the *T*-theorem to the *S4* theorem is known as "exportation."

[6] This line of criticism is taken from Richard Routley, *Relevant Logics and Their Rivals,* I, p. 11.

The reason exportation works in the disputed theorem of *S4* is that the conjuncts are modal sentences, equivalent to $\Box(\alpha \supset \beta)$ and $\Box(\beta \supset \gamma)$. This is what allowed reiteration into two restricted scope lines at lines 4 and 5 of the derivation of $(\alpha \multimap \beta) \multimap ((\beta \multimap \gamma) \multimap (\alpha \multimap \gamma))$. So Lewis would have us reject SR-$\multimap$(4) and SR-$\Box$(4). The question is whether there is a principled reason for so doing.

If the reason is that it blocks objectionable cases of exportation, then a case must be made for why those cases are objectionable, while others which are very similar are not. System *T* blocks exportation when one of the conjucts is non-modal, as with $\vDash_T (\alpha \wedge (\alpha \multimap \beta)) \multimap \beta$, but $\nvDash_T \alpha \multimap ((\alpha \multimap \beta) \multimap \beta)$. However, it allows exportation in the very similar case, $\vDash_T (\Box\alpha \wedge (\alpha \multimap \Box\beta)) \multimap \Box\beta$, and $\vDash_T \Box\alpha \multimap ((\alpha \multimap \beta) \multimap \Box\beta)$. Here, a necessity-sentence strictly implies that an implication implies a necessity-sentence. This is the same format as the result of exportation that Lewis rejected, $\Box(\alpha \supset \beta) \multimap ((\beta \multimap \gamma) \multimap \Box(\alpha \supset \gamma))$. The difference between these two types of exportation is very fine. It is so difficult to articulate that we might be best served to suppress it, as *S4* does.

4.3 Deontic Logic

Transitivity of accessibility seems undesirable in deontic logic in general. Suppose we are working with a notion of legal obligation, where the accessible worlds represent situations in which the law is fulfilled. Suppose further that we begin at the actual world and consider a world which is ideal with respect to the law. At that world, there may be other laws, generating their own ideal worlds. We would not want to say that we, in the actual world, are obligated by the laws passed in the ideal world. But we would be if accessibility were transitive.

Perhaps the situation is different on an absolutist view of moral obligation, such as a divine command theory or the Kantian theory of the "categorical imperative." The notion of absolute obligation in such a theory would be similar to that of logical necessity, and as such it would seem to require a system at least as strong as *S4*. A conditional notion of obligation, for which a weaker system such as *D* is more suitable, might apply to what Kant called a "hypothetical imperative."[7]

4.4 Doxastic Logic

The situation in doxastic and epistemic logic is controversial. When we believe something, do we believe that we believe it? When we know something, do we know that we know it? Affirmative answers to these questions are known as the BB and KK theses, respectively. Many philosophers embrace these theses and many have tried to refute them.

A problem can be seen from the standpoint of belief and carried over to knowledge. We often accuse people of being in "denial", in a state when they refuse to acknowledge what they actually believe, and hence do not believe that they believe it. Or take an unreflective person who does not even think about whether he has beliefs or not. Again, it would seem that such a person does not believe that he believes what he does.

But it must be kept in mind that even interpreting the *K*-systems as doxastic systems requires a good deal of idealization about the believer. The believer must, for example, believe all the valid sentences of the logic. It follows from closure that if $\vDash_{KI} \alpha$, then $\vDash_{KI} \mathrm{B}_{s,t}\alpha$. Moreover, another application of closure yields this further conditional: if $\vDash_{KI} \mathrm{B}_{s,t}\alpha$, then $\vDash_{KI} \mathrm{B}_{s,t}\mathrm{B}_{s,t}\alpha$.

So there is already a BB thesis with respect to valid sentences. If we bestow this degree of logical sainthood on the believer, then it may seem a small step to allow the BB thesis in general. Our logical saint would hardly be in denial or unreflective, and we might as well assume that he has complete command of what his beliefs are. Also, some more realistic accounts of belief, especially those which require reflective commitment to the truth of what is believed, are amenable to the BB thesis.

[7] Immanuel Kant, *Fundamental Principles of the Metaphysics of Morals*, 1785, First Section.

It is common to draw a distinction between "occurrent" and "dispositional" senses of belief. A belief is occurrent when what is believed is before the mind. A belief is dispositional if it need not be before the mind when believed. One instead has a disposition to assent (for example), when the content of belief comes before the mind. It may be that human beings can have a disposition assent to the proposition that they have a disposition to assent to something. And it may be that human being can occurrently believe that they are occurrently believing. But humans would quickly run out of resources if these higher-order beliefs, dispositional or occurrent, were to be iterated to any great extent. A "logical saint," however, could be taken as having an infinite capacity of belief-iteration, whether the beliefs are understood occurrently or dispositionally.

4.5 Epistemic Logic

The situation in epistemic logic is more complicated. Even if the BB thesis is granted, there are other conditions besides belief which must be met for a person to have knowledge. For example, it is widely held that the belief must be justified, warranted, or evident if it is to be an item of knowledge. In that case, if accessibility is transitive, one must be justified that one is justified, etc. On some theories of knowledge, especially "coherence" theories, this requirement seems natural.[8]

On the other hand, some theories of knowledge, such as theories which understand knowledge as the elimination of relevant alternatives, do not admit the KK thesis. Recall that one can understand the accessibility condition as defining a set of epistemically alternative situations. In order to know that α, α must be true in all those alternatives. But it need not be true in other alternatives which are not "relevant".

In the example given in a previous module, a person may know that he sees a zebra in the zoo, because he sees one in all situations relevantly like the current situation. A change in the angle of viewing, for example, would not make any difference. He would still be looking at a zebra. Suppose, however, that in an alternative situation in which he is the victim of an elaborate hoax, he does not see the zebra, but instead sees a mule painted to look like a zebra. He cannot distinguish this alternative from one in which he sees a zebra. But this does not matter so long as the alternative is not relevant.

It may, for some reason, be the case that in a relevant alternative some other alternative would become relevant. Perhaps if the subject moved five feet to the right he would hear someone saying that a prankster was at work in the zoo, fooling patrons by replacing zebras with painted mules. In that case, it might be that the hoax alternative is relevant. If transitivity holds for accessibility, this alternative would become relevant in the situation where the person does not hear about the hoax.

4.6 Temporal Logic

For the logic of futurity, transitivity of accessibility appears to be warranted. If at a time $\mathbf{t}$ α holds at all future times, then at any future time, α holds at all times in its future, since they are also in the future with respect to $\mathbf{t}$. So it is the case that at all times later than $\mathbf{t}$, α holds at the times in their futures. Similar remarks hold to claims about what holds at all past times.

[8] See G. J. Mattey "Lehrer's Personal Coherentism and the KK Thesis", *Philosophical Studies* 43 (1983), pp. 423-438.