Module 10
B and Equivalent Systems

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May 1, 2007

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We saw earlier that the T-systems yield the entailment of $\Diamond\alpha$ from $\{\alpha\}$, and that this seemed a desirable result for some of the applications we have been discussing. The B-systems take this a step further with its characteristic consequence of $\Box\Diamond\alpha$ from $\{\alpha\}$. The system is sometimes called the "Brouwerian" (or "Brouwersche") system, due to some similarities between its characteristic consequence-relations and an element of Brouwer's intuitionistic interpretation of mathematics, to be discussed in the last section. The system is obtained by adding a symmetry requirement for the accessibility relation to the semantical system *TI*.

The B-systems are extensions of the T-systems, and therefore are extensions of the K- and D-systems as well. Like the *S4*-systems, the B systems are not based directly on the K-systems. The T-systems' characteristic consequence-relations are independent of those of the B-systems, so there are systems with those consequence relations which are stronger than the K-systems but weaker than the B-systems. As with the other families of systems, we will discuss the semantical system first, then the derivational system, then, briefly, the axiom system, and finally applications.

1 The Semantical System *BI*

The axiom system *B* is built on *T* as a foundation, and correspondingly the semantical system *BI* is based on the semantical system *T*. Thus, the accessibility relation in a **BI**-frame is reflexive. That relation is also

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symmetrical:

R is **symmetrical** if and only if $(\Pi x)(\Pi y)(\mathbf{Rxy} \rightarrow \mathbf{Ryx})$.

If a world is accessible to another, then the other is accessible to it. Note that as with transitivity, this restriction is conditional. Because accessibility in the semantical system *BI* is serial (as a consequence of its reflexivity), the condition in the antecedent is always met for at least one world accessible to a given world **w**. Each world has at least one world accessible to it, and by symmetry, they are mutually accessible. So each world **w** stands in a symmetrical relation of accessibility to at least one world. This fact is guaranteed as well by reflexivity: every world is accessible to itself, in which case it trivially stands in an acessibility relation to itself: If **Rww**, then **Rww**.

Applied to frames, this means that if a world $\mathbf{w}_i$ that belongs to a frame is accessible to **w**, then **w** is accessible to $\mathbf{w}_i$.

We can define a *BI*-frame as a set $\langle \mathbf{W}, \mathbf{R} \rangle$, such that :

$(\Pi \mathbf{w})(\mathbf{w} \in \mathbf{W} \rightarrow \mathbf{Rww})$, and

$(\Pi \mathbf{w})(\Pi \mathbf{w}_i)((\mathbf{w} \in \mathbf{W} \bigwedge \mathbf{w}_i \in \mathbf{W}) \bigwedge \mathbf{Rww}_i) \rightarrow \mathbf{Rw}_i\mathbf{w})$.

The symmetry of the accessibility relation in *BI*-frames yields what will be called the characteristic consequence of the *B*-systems.

$\{\alpha\} \vDash_{BI} \Box\Diamond\alpha$.

The proof of the entailment can be given using a meta-logical derivation.

Sketch of a semantical proof that: $\{\alpha\} \vDash_{BI} \Box\Diamond\alpha$

1	$v_{\mathbf{I}}(\alpha, \mathbf{w}) = \mathbf{T}$		Assumption
2	$\mathbf{Rww}_1 \rightarrow \mathbf{Rw}_1\mathbf{w}$		Symmetry of **R**
3	$\mathbf{Rww}_1$		Assumption
4	$\mathbf{Rw}_1\mathbf{w}$		$2\ 3 \rightarrow E$
5	$\mathbf{Rw}_1\mathbf{w} \bigwedge v_{\mathbf{I}}(\alpha, \mathbf{w}) = \mathbf{T}$		$3\ 4 \bigwedge I$
6	$v_{\mathbf{I}}(\Diamond\alpha, \mathbf{w}_1) = \mathbf{T}$		5 **SR-**$\Diamond$
7	$\mathbf{Rww}_1 \rightarrow v_{\mathbf{I}}(\Diamond\alpha, \mathbf{w}_1) = \mathbf{T}$		$2\text{-}6 \rightarrow I$
8	$v_{\mathbf{I}}(\Box\Diamond\alpha, \mathbf{w}) = \mathbf{T}$		7 **SR-**$\Box$

Note that this derivation does not depend on any restrictions on **R** other than symmetry. So symmetry could be added to the semantical systems *KI* or *DI* rather than *TI*, to produce other semantical systems with the charateristic *B* consequence relation.

This reasoning can be illustrated with a modal truth-table.

w_1	$\rightleftarrows$	w_2
α		
T		
$\Diamond\alpha$		$\Diamond\alpha$
T		**T**
$\Box\Diamond\alpha$		
T		

2

Note that the inclusion of a value for '$\Diamond\alpha$' at w_1 is made because of reflexivity. In fact, since w_2 is an arbitrary world, w_2 could be w_1 itself, which is why this inclusion is not made in the meta-logical derivation.

All *TI4*-entailments (and hence all *DI*- and *KI*-entailments) are *BI*-entailments.

$$\{\gamma_1, \ldots, \gamma_n\} \vDash_{TI} \alpha \rightarrow \{\gamma_1, \ldots, \gamma_n\} \vDash_{BI} \alpha.$$

Since the class of *BI*-frames is a subset of the class of *TI*-frames, any entailment that holds in all *TI*-frames also holds in all *BI*-frames. So the semantical system *TI* is contained in the semantical system *BI*.

BI is a stronger system than *TI*, in that some *BI*-entailments are not *TI*-entailments, because some *TI*-frames are not *BI*-frames. So *BI* is an extension of *TI* (and hence of *DI* and *KI*. Specifically,

$$\{\alpha\} \nvDash_{TI} \Box\Diamond\alpha.$$

Proof. Let W in a frame Fr contain two worlds, w_1 and w_2, such that Rw_1w_1 Rw_1w_2 and Rw_2w_2. R is therefore reflexive, and so Fr is a *TI*-frame. Now let $\mathbf{v_I}(\alpha, w_1) = \mathbf{T}$ and $\mathbf{v_I}(\alpha, w_2) = \mathbf{F}$. If follows from **SR-$\Diamond$** that $\mathbf{v_I}(\Diamond\alpha, w_2) = \mathbf{F}$, since α is false at all accessible worlds, i.e., at w_2 itself. Therefore, $\mathbf{v_I}(\Box\Diamond\alpha, w_1) = \mathbf{F}$, by **SR-$\Box$**.

$$
\begin{array}{ccc}
\curvearrowleft & & \curvearrowleft \\
w_1 & \rightarrow & w_2 \\
\alpha & & \alpha \\
\hline
\mathbf{T} & & \mathbf{F} \\
\Diamond\alpha & & \Diamond\alpha \\
\hline
\mathbf{T} & & \mathbf{F} \\
\Box\Diamond\alpha & & \\
\hline
\mathbf{F} & &
\end{array}
$$

The semantical systems *BI* and *S4I* do not contain each other. The accessibility relation **R** may be transitive and reflexive without being symmetric and reflexive, and it may be symmetric and reflexive without being transitive and reflexive. (More generally, transitivity does not imply symmetry, nor does symmetry imply transitivity.)

The independence of the two semantical systems can be seen also from the fact that neither system supports the characteristic consequence relation of the other. The frame just given shows that the characteristic *BI* consequence does not hold in *S4I*. The accessibility relation in that frame is trivially transitive, since the condition that triggers the additional "pass-through" accessibility relation (so to speak) is not satisifed.

For the failure of the characteristic *S4I* consequence in a *BI*-frame, let the frame have three worlds, and let Rw_1w_1, Rw_2w_2, Rw_3w_3, Rw_1w_2, Rw_2w_3, Rw_2w_1, Rw_3w_2. In this frame, accessibility is symmetric and reflexive. Now let α be true at w_1 and w_2. Since α is true at all worlds accessible to w_1, $\Box\alpha$ is true at w_1. Now let α be false at w_3. The $\Box\alpha$ is false at w_2, in which case $\Box\Box\alpha$ is false at w_1.

$$
\begin{array}{ccccc}
\curvearrowleft & & \curvearrowleft & & \curvearrowleft \\
w_1 & \leftrightarrows & w_2 & \leftrightarrows & w_3 \\
\alpha & & \alpha & & \alpha \\
\hline
\mathbf{T} & & \mathbf{T} & & \mathbf{F} \\
& & \Box\alpha & & \\
\hline
& & \mathbf{F} & & \\
\Box\Box\alpha & & & & \\
\hline
\mathbf{F} & & & &
\end{array}
$$

2   The Derivational System *BD*

Just as the semantical system for *BI* contains the semantical rules inherited from *KI*, *DI*, and *TI*, the derivational system inherits the derivational rules from these systems. We will give an additional rule that generates the effects of symmetry of accessibility. It will be a special version of Strict Reiteration, SR (B), which allows that when α occurs, $\Diamond\alpha$ may be "reiterated" across a single restricted scope line.

This rule is unique among those based on *KD*, because the "reiteration" introduces a new operator altogether within the strict scope line, rather than removing one.[1] This is an expedient needed because of the incongruity between symmetry and the mechanics of Fitch-style natural deductions. When we introduce a strict scope line to the right, it indicates a world accessible to the current world. To indicate that the current world is accessible to another world, we would have to write a line to the left, which is not feasible in the Fitch system. (On the other hand, with the modal truth-tables, we could introduce an arrow pointing to the left.)

So we write within the restricted scope line a sentence that is the *effect* of the current world being accessible to that world. Because α is true at the current world, $\Diamond\alpha$ is true at any world to which the current world is accessible. So that is what we write down to the right of a strict scope line.[2]

Strict Reiteration (B)

	α	Already derived
$\Box$	$\vdots$	
	$\Diamond\alpha$	SR (B)

The rule is sound. If α is true at a world, then $\Diamond\alpha$, is true at an arbitrary accessible world, since by symmetry the current world is accessible from it.

We assert without proof that the derivational system *BD* resulting from adding this rule to the derivational rules of *TD* is complete with respect to the semantical system *BI*. We also assert without proof that the derivational system is sound. This claim can be motivated by the way in which a derivation mirrors some of the semantical reasoning used in the meta-logical derivation above. (It does not mirror all the reasoning, due to the limitation noted above of Fitch-style systems.)

To prove: $\{\alpha\} \vdash_{BD} \Box\Diamond\alpha$

1	α	Assumption
2	$\Box$ $\;\Diamond\alpha$	1 SR (B)
3	$\Box\Diamond\alpha$	2 $\Box$ I

For systems with '$\Box$' as primitive and '$\Diamond$' as derived, we can amend the strict reiteration rule using Duality. We begin with $\sim\alpha$ instead of α and strictly "reiterate" $\sim\Box\alpha$

Strict Reiteration-$\sim\Box$ (B)

	$\sim\alpha$	Already derived
$\Box$	$\vdots$	
	$\sim\Box\alpha$	SR-$\sim\Box$ (B)

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[1]This also occurs, for similar reasons, with the derivational rules needed to derive the axioms of K_t, as described in Module 6.

[2]This is similar to the alternative rule for *S4*, SR(4)F, which allows $\Box\alpha$ to be written down across one restricted scope line when $\Box\alpha$ occurs. SR(4)F simulates the effect of a third world at which α is written down.

This is easily seen to be a derived rule given the original system with SR (B).

Strict Reiteration for '~□' (B) as a derived rule

$$
\begin{array}{ll}
\sim\alpha & \text{Already derived} \\
\cdot & \\
\cdot & \\
\cdot & \\
\Diamond\sim\alpha & \text{SR (B)} \\
\sim\Box\alpha & \text{Duality}
\end{array}
$$

With this rule in hand, we can prove that $\{\alpha\} \vdash_{BD} \Box\sim\Box\sim\alpha$, which is the characteristic *BD* consequence in a syntax without the '$\Diamond$' operator.

To prove: $\{\alpha\} \vdash_{BD} \Box\sim\Box\sim\alpha$

$$
\begin{array}{lll}
1 & \alpha & \text{Assumption} \\
2 & \sim\sim\alpha & \text{1 Double Negation} \\
3 & \sim\Box\sim\alpha & \text{2 SR-}\sim\Box\text{ (B)} \\
4 & \Box\sim\Box\sim\alpha & \text{2 3 }\Box\text{ I}
\end{array}
$$

For systems with '$\Diamond$' alone as primitive, we can use the rule of Strict Reiteration (B) in conjunction with $\sim\Diamond$ Introduction to derive the dual of $\Box\Diamond\alpha$, i.e. $\sim\Diamond\sim\Diamond\alpha$, from $\{\alpha\}$.

To prove: $\{\alpha\} \vdash_{BD} \sim\Diamond\sim\Diamond\alpha$

$$
\begin{array}{lll}
1 & \alpha & \text{Assumption} \\
2 & \Diamond\alpha & \text{1 SR (B)} \\
3 & \sim\sim\Diamond\alpha & \text{2 Double Negation} \\
4 & \sim\Diamond\sim\Diamond\alpha & \text{2-3 }\sim\Diamond\text{ I}
\end{array}
$$

3   The Axiom System *B*

The axiom system *B* is obtained by adding to the axiom schemata of *T* the further axiom schema:

$\vdash_B \alpha \supset \Box\Diamond\alpha.$

This axiom is clearly valid in *BI*, by the same reasoning that was used to validate the corresponding semantical entailment.

4   Applications of the *B*-Systems

Symmetry of accessibility makes for odd consequences. For this reason, the **B**-systems only seem suitable for alethic applications which call for even stronger systems that contain them.

4.1 Alethic Modal Logic

The *B*-systems take us closer to representing the logical modalities. We have already remarked that if possibility is viewed logically, what is the case at a world should possibly be the case at that world as well. This what the *T*-systems yield. Assuming a uniform set of "laws of logic," it seems reasonable to take this a step further and say that when something is the case at a world , it is logically necessary that it be logically possible. That is, when something α is the case, it follows from the laws of logic that α is consistent with the laws of logic.

For hypothetical modalities, the symmetry restriction on accessibility may or may not be wanted, depending on the details of the condition accessibility is supposed to represent. Consider Hughes and Cresswell's example of conceivability, discussed in the context of the *S4*-systems. We may conceive a situation which contains individuals who cannot conceive of our situation. So from their standpoint, our world is not a possible one. On the other hand, we might want to represent conceivable worlds as containing individuals (perhaps even ourselves) who can conceive of us, in which case we would want accessibility to be symmetrical.

If the condition is a set of laws of nature, the considerations noted in the last module may apply. On the one hand, it may be that exactly the same laws hold at all accessible worlds, in which case symmetry seems appropriate. On the other hand, there may be some phenomena in an accessible world that are governed by additional laws of nature. Since condition holding for the accessible world is expanded, it may not be thought to apply appropriately to the home world, which is what symmetry would require.

4.2 Conditional Logic

As far as a logic of implication is concerned we have the result that $\vDash_{BI} \alpha \prec \Box\Diamond\alpha$. If we regard strict implication as representing a notion of logical implication the considerations about logical modalities from the last section apply here. The validity of $\alpha \prec \Box\Diamond\alpha$ may be thought desirable. If α is true, then it is a matter of logic that α is possibly true, and it is also a matter of logic that the possible truth of α is necessary. We may regard strict implication as holding locally, indicating only a connection of truth-values at accessible worlds. Then the considerations cited in the last two paragraphs show that we may or may not desire the result that at all accessible worlds where α holds, it is necessarily possible that α holds.

4.3 Deontic Logic

Because they contain the respective *T*-systems, the *B*-systems are too strong for deontic logic, regardless of their characteristic consequence-relation. But the characteristic consequence-relation for the *B*-systems is itself undesirable for a logic of obligation. If we deny that everything that is the case (at a world) is permissible, as is required by the *T*-systems, we would be even less inclined to admit a principle that states that everything that is the case is permissible as a matter of obligation.

4.4 Doxastic Logic

The same considerations apply to doxastic logic. We have denied when discussing the *T*-systems that what is the case is, as a matter of logic, compatible with what a "logical saint" believes at a time. There is all the more reason to deny that as a matter of logic, a "logical saint" believes of whatever is the case that it is compatible with what the person believes.

4.5   Epistemic Logic

Epistemic logics generally are built on the T-systems, so the problem for deontic and doxastic logics posed by the containment of the T-systems by the B-systems does not arise. Thus we have $\{\alpha\} \vDash_{BI} F_{s,t}\alpha$. But any item of knowledge seems to be as well a belief, and we have already seen that the B-systems' consequence relation is unsuitable for belief. For this reason, what the B-systems add to the T-systems is not suitable for a logic of knowledge. It implausibly requires that any truth is not only compatible with what the subject knows at the time, but is known to be compatible: $\{\alpha\} \vDash_{BI} K_{s,t} F_{s,t}\alpha$. A "logical saint" might know what holds as a matter of logic, but a contingent sentence α may or may not be compatible with what he knows, and this does not seem to be guaranteed by its mere truth.

4.6   Temporal Logic

A temporal logic of what is past and what is future would have to regard the past and future in a very odd way if it were to admit the characteristic consequence relation of the B-systems. Such a logic would be based on the T-systems, which have already been seen to be undesirable, since something's being the case now does not entail that it will be the case at some future time.

Of course, we could introduce new operators signifying what is the case "now and forever" and what is the case now or at some future time. Then the T-systems would be appropriate. What is now and forever the case is now the case, and what is now the case is the case now or at some time in the future. Still, we would not want to say that if α is now the case, that it now and forever is the case that it now is or will at some time be the case that α. In the B-systems, the original operators representing the "future" and the "past" (minus the reference to the present) would function symmetrically. A time in the future now is such that now is in its future, etc. This would undermine entirely the directional character of our notions of future and past.

4.7   The Brouwer Connection

Intuitionist mathematicians such as Brouwer were opposed to any mathematical proofs which employ the rule we here call Negation Elimination. If one assumes that the negation of some mathematical proposition is true and derives a contradiction, it would follow from Negation Elimination that the proposition itself is true. The intuitionists held that this would allow the proof of the existence of mathematical objects without any positive evidence that the object exists. This kind of reasoning is used in meta-logic as well. It is often "proved" that a derivation exists by denying that there is a derivation and showing that a contradiction follows from this denial.

If we were to remove Negation Elimination from our primitive rule-set, we would not be able to prove that α follows from $\sim\sim\alpha$. Thus for an "intuitionist logic" I, $\{\sim\sim\alpha\} \nvdash_I \alpha$. On the other hand, the converse consequence, $\{\alpha\} \vdash_I \sim\sim\alpha$, is allowed.

Consider Negation Introduction, according to which if the assumption of α leads to a contradiction, we may assert $\sim\alpha$. Here we might be tempted to say that what is really proved by the rule is that α is *impossible*, i.e., $\sim\Diamond\alpha$. We can, in fact, get this result in *KD*.

1	$\square$	α	Assumption
2		$\vdots$	
3		β	
4		$\sim\beta$	
5		$\sim\alpha$	Negation Introduction
6	$\sim\Diamond\alpha$		$\sim\Diamond$ I

The sentence-schema $\sim\Diamond\alpha$ can be symbolized with the impossibility operator '$\sim$.' Then the characteristic *B* consequence $\{\alpha\} \vdash_B \sim\Diamond\sim\Diamond\alpha$ becomes $\{\alpha\} \vdash_B \sim\sim\alpha$, which lines up with what is allowed for negation by the intuitionists. On the other hand, $\{\sim\Diamond\sim\Diamond\alpha\} \nvdash_B \alpha$, and thus $\{\sim\sim\alpha\} \nvdash_B \alpha$. This fact is shown by constructing a modal truth-table in which $\sim\sim\alpha$ is true and α is false at a world.

w_1		$\leftrightarrows$	w_2
$\sim\Diamond\sim\Diamond\alpha$	α		α
T	**F**		**T**

$\Diamond\sim\Diamond\alpha$	$\Diamond\alpha$		$\Diamond\alpha$
F	**T**		**T**
	$\sim\Diamond\alpha$		$\sim\Diamond\alpha$
	F		**F**

So, the behavior of the impossibility operator in the Brouwersche systems has an affinity with that of the negation operator in intuitionistic logic.[3]

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[3] See Hughes and Cresswell, *A New Introduction to Modal Logic*, pp. 70-71.