Module 18
Systems with a Unitary Domain of Possible Objects

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Systems with a unitary domain of possible objects are strong systems that can be built on the framework of Sentential Logic and Free Predicate Logic we have developed to this point. They sanction both the Barcan Consequences and the Converse Barcan Consequences, and hence allow the reversal of the pairs $\forall - \Box$ and $\exists - \Diamond$. The other two pairs work in only one direction, a fact which will be discussed below. A family of systems, the *FQ1-x* systems, is built on the platform of Free Predicate Logic and will be considered first here. A stronger family of systems, the *Q1-x* systems, is based on standard Predicate Logic and will be the last topic to be treated in this series of modules.

1 The *FQ1-x* Systems

In this section, we will give semantical rules for systems *FQ1I-x* and derivational rules for systems **FQ1D-x**.

1.1 The Semantical *FQ1I-x* Systems

The key semantical feature of the *FQ1I-x* systems is that there is only one domain for all the objects at all the possible worlds. As noted in the last module, the effect of a unitary domain is the consequence of both the symmetry of the accessibility and the Included In or Includes requirements on the domains at worlds. This is enough to yield the *FQ1-x* systems. Here we will simplify our treatment and begin with the assumption of a single domain serving all worlds.

Since the underlying logic is Free Predicate Logic, we will have to re-introduce the distinction between an inner and an outer domain—a distinction which in the *Q1RI-x* systems had been transposed to the difference between the domain of the world and the remaining objects in the domain of the frame. Since the unitary domain exhausts the objects existing at all the possible worlds, the outer domain can only consist of what are (from the standpoint of the frame), impossible objects. So if we want '*a*' to refer to a round square, we would still be able to assert that it is round, which might be symbolized as '*Fa*.'

The semantical rules for *FQ1I-x* are the same as those for *FPI*, only relativized to worlds.

Interpretations in *FQ1I-x*

$\mathbf{I} = \{\mathbf{D}, \mathbf{D}', \mathbf{v}\}$

$\mathbf{D} \neq \varnothing$

$\mathbf{D} \cap \mathbf{D}' = \varnothing$

$\mathbf{W} \neq \varnothing$

Special Semantical Rules for *FQ1I-x*

1. $\mathbf{v_I}(\mathbf{a}) \in \mathbf{D} \cup \mathbf{D}'$
2. $\mathbf{v_I}(\mathbf{u}) \in \mathbf{D} \cup \mathbf{D}'$
3. $(\Pi \mathbf{w}_i)(\Pi \mathbf{w}_j)((\mathbf{w}_i \in \mathbf{W} \wedge \mathbf{w}_j \in \mathbf{W}) \rightarrow (\mathbf{v_I}(\mathbf{t}, \mathbf{w}_i) = (\mathbf{v_I}(\mathbf{t}, \mathbf{w}_j))$
4. $\mathbf{v}(\mathbf{P}^n) \subseteq (\mathbf{D} \cup \mathbf{D}')^n$
5. $\mathbf{v_I}(\mathrm{E}) = \mathbf{D}$[1]
6. $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$; $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{F}$ if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.
7. $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$; $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{F}$ if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_{\mathrm{I}}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.
8. $\mathbf{v_I}(\mathbf{t}_i = \mathbf{t}_j, \mathbf{w}) = \mathbf{T}$ if and only if $\mathbf{v_I}(\mathbf{t}_i, \mathbf{w}) = \mathbf{v_I}(\mathbf{t}_j, \mathbf{w})$;
   $\mathbf{v_I}(\mathbf{t}_i = \mathbf{t}_j, \mathbf{w}) = \mathbf{F}$ if and only if $\mathbf{v_I}(\mathbf{t}_i) \neq \mathbf{v_I}(\mathbf{t}_j, \mathbf{w})$;

It should be obvious that the Barcan and Converse Barcan Consequences hold in this semantical system, since any reasoning involving the inclusion relation between domains of worlds holds for a system in which there is a single world. Here is a diagram that illustrates how a variation of the Barcan Consequence would be proved in *FQ1-S5*. Since there is no longer any need to appeal to the domains of worlds, so the diagram is more simple than that for *Q1RC-S5*.

$$
\begin{array}{ccc}
\mathbf{w}_1 & \xrightarrow{*} & \mathbf{w}_2 \\
\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & & \\
\hline
\mathbf{T} & & \\
 & & (\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) \\
\hline
 & & \mathbf{T} \\
 & & \alpha(\mathbf{u})\ [\Pi \mathbf{d}/\mathbf{u}] \\
\hline
 & & \mathbf{T} \\
\Box\alpha(\mathbf{u})\ [\Pi \mathbf{d}/\mathbf{u}] & & \\
\hline
\mathbf{T} & & \\
(\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u}) & & \\
\hline
\mathbf{T} & &
\end{array}
$$

The fact that both the Barcan and Convese Barcan Consequences hold in the semantical system means that sentences beginning with '$\forall - \Box$' are equivalent to those beginning with '$\Box - \forall$,' and likewise for the pairs '$\exists - \Diamond$' and '$\Diamond - \exists$.' This leaves two more pairs, each of which allows an entailment in only one direction. Thus we have:

$\{(\exists \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{FQ1I-x} \Box(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$.

$\{\Diamond(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{FQ1I-x} (\forall \mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})$.

The first result will be illustrated with a diagram.

w_1	$\leftrightarrows$	w_2
$\Diamond(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$		
T		
		$(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$
		T
		$\alpha(\mathbf{v}/\mathbf{u})$ $[\Pi\mathbf{d}/\mathbf{v}]$
		T
$\Diamond\alpha(\mathbf{v}/\mathbf{u})$ $[\Pi\mathbf{d}/\mathbf{v}]$		
T		
$(\forall \mathbf{x}\Diamond\alpha(\mathbf{x}/\mathbf{u})$		
T		

The second result is left as an exercise for the reader.

The following non-entailments were proved in the previous model using the underlying system *KI*, with an interpretation with a single domain, so they apply to the present systems as well. We shall now illustrate why they fail even if the underlying semantical system is *S5I*.

$\{\Box(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \nvDash_{FQ1I-S5} (\exists \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$.

$\{(\forall \mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})\} \nvDash_{FQ1I-S5} \Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$.

w_1	$\leftrightarrows$	w_2
$F : \langle 1 \rangle$		$F : \langle 2 \rangle$
$\Box(\exists x)Fx$		
T		
$(\exists x)Fx$		$(\exists x)Fx$
T		**T**
Fu $[1/u]$		Fu $[2/u]$
T		**T**
Fu $[2/u]$		Fu $[1/u]$
F		**F**
$\Box Fu$ $[1/u]$		
F		
$\Box Fu$ $[2/u]$		
F		
$(\exists x)\Box Fx$		
F		

Exercise. Explain why the counter-example works for the other non-entailment, and illustrate your reasoning with a diagram.

Summary of *FQ1I-x*-Systems
I=⟨**W**, **R**, **D**, **D′v**⟩
Semantical rules for '□,' '◊,' '⥽' analogous to those for *KI*
Semantical rules for '∀,' '∃' as for *FPI*
Conditions on **R** for system **x**
Rigid Designation

1.2 The Derivational *FQ1D-x* Systems

For the derivational systems *FQ1D-x*, we simply combine the rules for *Q1RCD-x* and *Q1RB-x*. If we were to drop the indices, then we would have to decide how to handle situations like the following.

Attempt to prove: $\{\Box(\forall x)Fx\} \vdash_{Q1RCD-K} (\forall x)\Box Fx$

1		$\Box(\forall x)Fx$	Assumption
2	u	$\Box(\forall x)Fx$	1 Reiteration
3	$\Box$	$(\forall x)Fx$	2 SR-□
4		$F\underline{v}$	3 ∀ E (*FQ1*)
5		$F\underline{v}$	6 BR ∀ (*FQ1*)
6		$\Box F\underline{v}$	2 3-7 □ I
7		$(\forall x)\Box Fx$	8 Misapplication of ∀ I (*FPI*)

We would then need some rule which tells us when we can take the final step. If we allowed unrestricted generalization, we would permit invalid inferences, such as the following.

Attempt to prove: $\{(\forall x)\Diamond Fx\} \vdash_{Q1RCD-K} \Diamond(\forall x)Fx$

1		$(\forall x)\Diamond Fx$	Assumption
2	u	$(\forall x)\Diamond Fx$	1 Reiteration
3		$F\underline{u}$	2 ∀ E
4	$\Box$	$F\underline{u}$	Strict Assumption
5	v	$F\underline{u}$	4 Reiteration
6		$(\forall x)Fx$	5 Missapplication of ∀ I
7		$\Diamond(\forall x)Fx$	2 4-6 W ◊ I
8		$\Diamond(\forall x)Fx$	6 BR *FQ1*

2 The *Q1-x* Systems

In this section, we will give semantical rules for systems *Q1I-x* and derivational rules for systems **Q1D-x**. These rules will be simplest of all.

2.1 The Semantical *Q1I-x* Systems

The difference between the systems *FQ1I-x* and *Q1I-x* lies in the absence of a distinction between inner and outer domains in the semantics. There is a single domain, *D*, in every frame. We shall require that every term (parameter or constant) designates an object in **D**.

Interpretations in *Q1I-x*

$\mathbf{I} = \{\mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{v}\}$

$\mathbf{D} \neq \varnothing$

$\mathbf{W} \neq \varnothing$

Special Semantical Rules for *Q1I-x*

1. $\mathbf{v_I}(\mathbf{a}) \in \mathbf{D}$
2. $\mathbf{v_I}(\mathbf{u}) \in \mathbf{D}$
3. $(\Pi \mathbf{w}_i)(\Pi \mathbf{w}_j)((\mathbf{w}_i \in \mathbf{W} \bigwedge \mathbf{w}_j \in \mathbf{W}) \to (\mathbf{v_I}(\mathbf{t}, \mathbf{w}_i) = (\mathbf{v_I}(\mathbf{t}, \mathbf{w}_j))$
4. $\mathbf{v}(\mathbf{P}^n) \subseteq \mathbf{D}^n$
5. $\mathbf{v_I}(\mathrm{E}) = \mathbf{D}^1$
6. $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$; $\mathbf{v_I}((\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{F}$ if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.
7. $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{T}$ (where $\mathbf{u}$ is free for $\mathbf{x}$ in $\alpha(\mathbf{x})$) if and only if for some $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{T}$; $\mathbf{v_I}((\exists \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}) = \mathbf{F}$ if and only if for all $\mathbf{d} \in \mathbf{D}$, $\mathbf{v}_\mathrm{I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}) = \mathbf{F}$.
8. $\mathbf{v_I}(\mathbf{t}_i = \mathbf{t}_j, \mathbf{w}) = \mathbf{T}$ if and only if $\mathbf{v_I}(\mathbf{t}_i, \mathbf{w}) = \mathbf{v_I}(\mathbf{t}_j, \mathbf{w})$;
   $\mathbf{v_I}(\mathbf{t}_i = \mathbf{t}_j, \mathbf{w}) = \mathbf{F}$ if and only if $\mathbf{v_I}(\mathbf{t}_i) \neq \mathbf{v_I}(\mathbf{t}_j, \mathbf{w})$;

This change in the semantics has important consequences for sentences containing constants. It restores the *PI* validity (in systems as weak as *Q1I-K*) of $(\exists \mathbf{x})\mathbf{x} = \mathbf{a}$. Because of Closure, we have $vDash_{Q1I-K}\Box(\exists \mathbf{x})\mathbf{x} = \mathbf{a}$. We also have $vDash_{Q1I-K}(\exists \mathbf{x})\Box \mathbf{x} = \mathbf{a}$.

Summary of *1Q1I-x*-Systems
I=⟨**W**, **R**, **Dv**⟩
Semantical rules for '□,' '◊,' '⥽' analogous to those for *KI*
Semantical rules for '∀,' '∃' with parameters as for *FPI*
Semantical rules for '∀,' '∃' with constants as for *PI* Conditions on **R** for system **x**
Rigid Designation

2.2 The Derivational *Q1D-x* Systems

We will take over all the derivational rules for the *FQ1RD-x* systems, except that, to reflect a unitary domain, we can now relax the rules for the instantiation of univerally and existentially quantified formulas and use the rules for *PD*, though limited to constants.

Universal Elimination (*Q1D*)

$$\begin{array}{|ll}
(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \text{Already Derived} \\
\vdots & \\
\alpha(\mathbf{a}/\mathbf{u}) & \forall\ \mathrm{I}
\end{array}$$

Existential Introduction (*Q1D*)

$$\begin{array}{|ll}
\alpha(\mathbf{a}/\mathbf{u}) & \text{Already Derived} \\
\vdots & \\
(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & \exists\ \mathrm{I}
\end{array}$$

No longer must we have ‘E**a**’ to instantiate from ‘$(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$’ to ‘$\alpha(\mathbf{a}/\mathbf{u})$.’ Nor is it needed for the generalization from ‘$\alpha(\mathbf{a}/\mathbf{u})$’ to ‘$(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$.’ This change does not affect the rules involving parameters. When the rules of Universal Instantiation and Existential Generalization are applied to formulas by virtue of their constants, there is need for a barrier.

Using the first of these relaxed rules, we can prove as theorems what before were shown to be valid:

To prove: $\vdash_{Q1D-K} \Box(\exists x)x = a$

$$\begin{array}{l|l|ll}
1 & \Box^1 & a = a & = \mathrm{I} \\
2 & & (\exists x)x = a & 1\ \exists\ \mathrm{I}\ (\textit{Q1D-x}) \\
\hline
3 & \multicolumn{2}{l}{\Box(\exists x)x = a} & \text{1-2}\ \Box\ \mathrm{I}
\end{array}$$

To prove: $\vdash_{Q1D-K} (\exists x)\Box x = a$

$$\begin{array}{l|l|ll}
1 & \Box^1 & a = a & = \mathrm{I} \\
\hline
2 & \multicolumn{2}{l}{\Box a = a} & 1\ \Box\ \mathrm{I} \\
3 & \multicolumn{2}{l}{(\exists x)\Box x = a} & 2\ \exists\ \mathrm{I}\ (\textit{Q1D})
\end{array}$$