Module 17
Systems with Nested Domains

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Contents

The *Q1R-x* systems are very weak. They allow for the representation of sentences about non-existent objects and they are neutral with respect to the relation between modalities *de re* and *de dicto*. It is not difficult, however, to generate systems stronger than the *Q1R-x* systems which make these connections on the basis of their semantical rules. What is needed is a stipulation which correlates the domains of worlds with the relation of accessibility.

We can require either that the domain of the home world *include* (or be a *superset of*) the domains of all the accessible worlds or that the domain of the home world be *included in* (or a *subset of*) the domains of each accessible world.[1] Alternatively, one may say that the domain of the accessible worlds are "nested" in that of the home world, or that the domain of the home world is nested in the domains of the accessible

[1]Hughes and Cresswell refer to the second of these requirements as "the" inclusion requirement, *A New Introduction to Modal Logic*, p. 275. The semantical system they present containing this requirement, *QPLI*, is based on standard Predicate Logic and hence differs from the current systems.

worlds. In the former case, the Barcan Consequences hold and in the latter the Converse Barcan Consequences hold. So we can name the systems the *Q1RB-x* and *Q1RC-x* systems, respectively. In this module, we will first give both semantical systems *Q1RBI-x* and *Q1RCI-x*. Then we will lay out the somewhat tricky derivational *Q1RBD-x* and *Q1RCD-x* systems.

1 The Semantical Systems *Q1RBI-x* and *Q1RCI-x*

The distinctive feature of the *Q1RBI-x* systems is that the Barcan Consequences are semantical entailments with the systems:

Barcan Consequences
$\{(\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RBI-x} \Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$,
$\{\Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RBI-x} (\exists \mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})$.

The *Q1RCI-x* systems are distinguished by yielding the Converse Barcan Consequences as semantical entailments:

Converse Barcan Consequences
$\{\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RCI-x} (\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$,
$\{(\exists \mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RCI-x} \Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$.

1.1 Nested Domains

The *Q1RBI-x* semantical systems are based on the requirement that the domain of the home world *includes* the domains of each of the accessible worlds. In the language of set theory, the domain of the home world is a *superset* of the domain of each accessible. Each member of the domain of each accessible world is a member of the domain of the home world. To put the matter another way, the domain of the home world may *contract* as we move to accessible worlds, but it never "expands" to contain more members than does the domain of the home world.

The semantical systems are based on adding to the semantical systems *Q1R-x* the following proviso:

Includes

$\mathbf{Rw}_i\mathbf{w}_j \rightarrow \mathbf{D}^{\mathbf{w}_i} \supseteq \mathbf{D}^{\mathbf{w}_j}$

To generate the converse results, we need only reverse the inclusion (or nesting) condition so that the domain of the home world is *included in* the domain of each of the accessible worlds. Under this condition, the domains of the accessible worlds may *expand* but may not "contract" in the sense of losing members in the transition from the home world to an accessible world.

Included in

$\mathbf{Rw}_i\mathbf{w}_j \rightarrow \mathbf{D}^{\mathbf{w}_i} \subseteq \mathbf{D}^{\mathbf{w}_j}$

We can say immediately that the systems *Q1RBI-x* and *Q1RCI-x* are contained in the systems *Q1RI-x*. Whatever is valid in the systems with no restrictions on the domain will remain valid if the restriction Included In is added. Moreover, if both conditions hold, we generate systems stronger than both: the *Q1-x* systems to be examined in the next module.

In semantical systems where accessibility is symmetrical, both nesting conditions must hold together. This can be proved meta-logically. Here, the result is that the relation being Included In implies the relation Includes. The converse result can be proved easily and is left as an exercise.

1	$\mathbf{Rw_2w_1} \to \mathbf{D^{w_2}} \subseteq \mathbf{D^{w_1}}$	Included In
2	$\mathbf{Rw_1w_2} \to \mathbf{Rw_2w_1}$	Symmetry
3	$\quad \mathbf{Rw_1w_2}$	Assumption
4	$\quad \mathbf{Rw_2w_1}$	2 3 $\to$ E
5	$\quad \mathbf{D^{w_2}} \subseteq \mathbf{D^{w_1}}$	1 4 $\to$ E
6	$\mathbf{Rw_1w_2} \to \mathbf{D^{w_2}} \subseteq \mathbf{D^{w_1}}$	1-5 $\to$ I
7	$\mathbf{Rw_1w_2} \to \mathbf{D^{w_1}} \supseteq \mathbf{D^{w_2}}$	6 Definition

1.2 The $\forall - \Box$ Form of the Barcan Consequences

We will begin our examination of these semantical systems by proving informally the version of the Barcan Consequences involving the universal quantifier and the necessitation operator. Suppose that $(\forall\mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$ is true at an arbitrary world $\mathbf{w}^1$ for some interpretation $\mathbf{I}$. Then all the members of the domain $\mathbf{D^{w_1}}$ meet the condition $\Box\alpha(\mathbf{u})$ when assigned to '*u*.. So at an arbitrary accessible world $\mathbf{w}_2$, they meet the condition $\alpha(\mathbf{u})$. Because all the members of the domain $\mathbf{D^{w_2}}$ are members of the domain $\mathbf{D^{w_1}}$, the condition holds for all of them, in which case, $(\forall\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ is true at $\mathbf{w}_2$. In that case, $\Box(\forall\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ is true at $\mathbf{w}_1$.

The following is a formal meta-logical derivation for the underlying modal system *KI* embodying this reasoning.

To prove: $\{(\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RBI-K} \Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$

1	$\mathbf{Rw}_1\mathbf{w}_2 \rightarrow \mathbf{D}^{\mathbf{w}_1} \supseteq \mathbf{D}^{\mathbf{w}_2}$	Includes
2	$\mathbf{v_I}((\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	Assumption
3	$\mathbf{Rw}_1\mathbf{w}_2$	Assumption
4	$\mathbf{d}_1 \in \mathbf{D}^{\mathbf{w}_2}$	Assumption
5	$\mathbf{D}^{\mathbf{w}_1} \supseteq \mathbf{D}^{\mathbf{w}_2}$	1 3 $\rightarrow$ E
6	$\mathbf{d}_1 \in \mathbf{D}^{\mathbf{w}_1}$	4 5 Set Theory
7	$(\Pi \mathbf{d}_i)(\mathbf{d}_i \in \mathbf{D}^{\mathbf{w}_1} \rightarrow \mathbf{v_I}[\mathbf{d}_i/\mathbf{u}](\Box\alpha(\mathbf{u}), \mathbf{w}_1 = \mathbf{T})$	2 **SR**-$\forall$
8	$\mathbf{d}_1 \in \mathbf{D}^{\mathbf{w}_1} \rightarrow \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\Box\alpha(\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	7 $\forall$ E
9	$\mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\Box\alpha(\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	4 8 $\rightarrow$ E
10	$(\Pi \mathbf{w}_i)(\mathbf{Rw}_1\mathbf{w}_i \rightarrow, \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_1 = \mathbf{T})$	9 **SR**-$\Box$
11	$\mathbf{Rw}_1\mathbf{w}_2 \rightarrow, \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2 = \mathbf{T}$	10 $\forall$ E
12	$\mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	3 11 $\rightarrow$ E
13	$\mathbf{d}_1 \in \mathbf{D}^{\mathbf{w}_2} \rightarrow \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	4-12 $\rightarrow$ I
14	$(\Pi \mathbf{d}_i)(\mathbf{d}_i \in \mathbf{D}^{\mathbf{w}_2} \rightarrow \mathbf{v_I}[\mathbf{d}_i/\mathbf{u}](\alpha(\mathbf{u})), \mathbf{w}_2) = \mathbf{T}$	13 Π I
15	$\mathbf{v_I}(\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_2) = \mathbf{T}$	14 **SR**-$\forall$
16	$\mathbf{Rw}_1\mathbf{w}_2 \rightarrow \mathbf{v_I}(\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	3-15 $\rightarrow$ I
17	$(\Pi \mathbf{w}_i)(\mathbf{Rw}_1\mathbf{w}_i \rightarrow \mathbf{v_I}(\forall \mathbf{x})(\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w_i}) = \mathbf{T})$	16 $\forall$ I
18	$\mathbf{v_I}(\Box(\forall \mathbf{x}(\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_2) = \mathbf{T}$	17 **SR**-$\Box$
19	$\mathbf{v_I}(\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_1) = \mathbf{T} \rightarrow \mathbf{v_I}(\forall \mathbf{x})(\Box\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_1 = \mathbf{T}$	1-18 $\rightarrow$ I
20	$(\forall \mathbf{w}_i)(\mathbf{v_I}(\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_i = \mathbf{T} \rightarrow \mathbf{v_I}(\forall \mathbf{x})(\Box\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_i = \mathbf{T})$	19 $\forall$ I

This reasoning can be illustrated in diagrammatic form.

$\mathbf{w}_1$	$\xrightarrow{*}$	$\mathbf{w}_2$
$\mathbf{D}^{\mathbf{w}_1}$	$\supseteq$	$\mathbf{D}^{\mathbf{w}_2}$
$(\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$		
T		
$\Box\alpha(\mathbf{u})\ [\Pi\mathbf{d}^1/\mathbf{u}]$		
T		
		$\alpha(\mathbf{u})[\Pi\mathbf{d}^1/\mathbf{u}]$
		T
		$\alpha(\mathbf{u})[\Pi\mathbf{d}^2/\mathbf{u}]$
		T
		$(\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$
		T
$\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$		
T		

1.3 The $\Box - \forall$ Form of the Converse Barcan Consequences

Next we turn to the '$\Box - \forall$' version of the Converse Barcan Consequences. Suppose that $\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ is true at a world. Then at all accessible worlds, the condition α holds of all the objects in those worlds. But all the objects in the home world are, by Included In, in the domains of the accessible worlds. So for each of them the condition α holds at all the accessible worlds, and hence for each of them, the condition α holds necessarily.

Here is a formal meta-logical proof that the first consequence holds in *Q1RC-K*. After assuming that a sentence of the form $\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$ is true on an aribtrary interpretation at an arbitrary world, we will assume that $\mathbf{d}$ is in the domain of that world. Based on that assumption, we will prove that an arbitrary variant of the valuation function makes $\Box\alpha(\mathbf{x}/\mathbf{u})$ true at the original world, in which case the sentence of the form $(\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$ is true at that world.

To prove: $\{\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RC-K} (\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$

1	$\mathbf{Rw}_1\mathbf{w}_2 \rightarrow \mathbf{D}^{\mathbf{w}_1} \subseteq \mathbf{D}^{\mathbf{w}_2}$	Included In
2	$\mathbf{v_I}(\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_1 = \mathbf{T}$	Assumption
3	$\mathbf{d}_1 \in \mathbf{D}^{\mathbf{w}_1}$	Assumption
4	$(\Pi \mathbf{w}_i)(\mathbf{Rw}_1\mathbf{w}_i \rightarrow \mathbf{v_I}((\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_1) = \mathbf{T})$	2 **SR**-$\Box$
5	$\mathbf{Rw}_1\mathbf{w}_2 \rightarrow \mathbf{v_I}((\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_2) = \mathbf{T})$	4 Π E
6	$\mathbf{Rw}_1\mathbf{w}_2$	Assumption
7	$\mathbf{v_I}((\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_2) = \mathbf{T})$	5 6 $\rightarrow$ E
8	$(\Pi \mathbf{d}_i)(\mathbf{d}_i \in \mathbf{D}^{\mathbf{w}_2} \rightarrow \mathbf{v_I}[\mathbf{d_i}/\mathbf{u}](\alpha(\mathbf{u})), \mathbf{w}_1) = \mathbf{T})$	7 **SR**-$\forall$
9	$\mathbf{d}_1 \in \mathbf{D}^{\mathbf{w}_2} \rightarrow \mathbf{v_I}[\mathbf{d_1}/\mathbf{u}](\alpha(\mathbf{u})), \mathbf{w}_2) = \mathbf{T}$	8 $\forall$ E
10	$\mathbf{D}^{\mathbf{w}_1} \subseteq \mathbf{D}^{\mathbf{w}_2}$	Included in
11	$\mathbf{d}_1 \in \mathbf{D}^{\mathbf{w}_2}$	3 10 Set Theory
12	$\mathbf{v_I}[\mathbf{d_1}/\mathbf{u}](\alpha(\mathbf{u})), \mathbf{w}_2) = \mathbf{T}$	9 11 $\rightarrow$ E
13	$\mathbf{Rw}_1\mathbf{w}_2 \rightarrow \mathbf{v_I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u})), \mathbf{w}_2) = \mathbf{T}$	6-12 $\rightarrow$ I
14	$(\Pi \mathbf{w}_i)(\mathbf{Rw}_1\mathbf{w}_i \rightarrow \mathbf{v_I}[\mathbf{d}/\mathbf{u}](\alpha(\mathbf{u})), \mathbf{w}_2) = \mathbf{T})$	13 $\forall$ I
15	$\mathbf{v_I}[\mathbf{d}/\mathbf{u}](\Box\alpha(\mathbf{u})), \mathbf{w}_1 = \mathbf{T}$	14 **SR**-$\Box$
16	$\mathbf{d}_1 \in \mathbf{D}^{\mathbf{w}_1} \rightarrow \mathbf{v_I}[\mathbf{d}/\mathbf{u}](\Box\alpha(\mathbf{u})), \mathbf{w}_1) = \mathbf{T}$	3-15 $\forall$ I
17	$\mathbf{v_I}(\forall \mathbf{x})(\Box\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_1) = \mathbf{T}$	16 **SR**-$\forall$
18	$\mathbf{v_I}(\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_1) = \mathbf{T}) \rightarrow \mathbf{v_I}(\forall \mathbf{x})(\Box\alpha(\mathbf{x}/\mathbf{u})), \mathbf{w}_1) = \mathbf{T}$	2-17 $\rightarrow$ I

This reasoning can be illustrated in diagrammatic form.

$\mathbf{w}_1$	$\xrightarrow{*}$	$\mathbf{w}_2$
$\mathbf{D}^{\mathbf{w}_1}$	$\subseteq$	$\mathbf{D}^{\mathbf{w}_2}$
$\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$		
$\mathbf{T}$		
		$(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$
		$\mathbf{T}$
		$\alpha(\mathbf{u})\ [\Pi \mathbf{d}^2/\mathbf{u}]$
		$\mathbf{T}$
		$\alpha(\mathbf{u})\ [\Pi \mathbf{d}^1/\mathbf{u}]$
		$\mathbf{T}$
$\Box\alpha(\mathbf{u})\ [\Pi \mathbf{d}^1/\mathbf{u}]$		
$\mathbf{T}$		
$(\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$		
$\mathbf{T}$		

1.4 The $\Diamond - \exists$ Form of the Barcan Consequences

Recall that the second semantical version of the Barcan Consequences is: $\{\Diamond(\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RBI-x} (\exists\mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})$. We will first discuss the entailment informally, then go on to give a formal semantical proof of it. Suppose that in some world accessible to a home world, something meets the condition α. Then by the Inclusion condition, that object is an object at the home world. So it is possible, for some object at the home world, that it meet the condition α. Hence, there is an object at the home world which possibly meets the condition α.

To prove: $\{\Diamond(\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RBI-K} (\exists\mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})$

1	$\mathbf{Rw}_1\mathbf{w}_2 \rightarrow \mathbf{D}^{\mathbf{w}_1} \supseteq \mathbf{D}^{\mathbf{w}_2}$	Includes
2	$\mathbf{v_I}(\Diamond(\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	Assumption
3	$(\Sigma\mathbf{w}_i)(\mathbf{Rw}_1\mathbf{w}_i \bigwedge \mathbf{v_I}((\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_i) = \mathbf{T}$	2 **SR**-$\Diamond$
4	$\mathbf{Rw}_1\mathbf{w}_2 \bigwedge \mathbf{v_I}((\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	Assumption
5	$\mathbf{Rw}_1\mathbf{w}_2$	4 $\bigwedge$ E
6	$\mathbf{v_I}((\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	4 $\bigwedge$ E
7	$(\Sigma\mathbf{d}_i)(\mathbf{d}_i \in \mathbf{D}^2 \bigwedge \mathbf{v_I}[\mathbf{d}_i/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T})$	8 **SR**-$\exists$
8	$\mathbf{d}_1 \in \mathbf{D}^2 \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	Assumption
9	$\mathbf{d}_1 \in \mathbf{D}^2$	8 $\bigwedge$ E
10	$\mathbf{D}^{\mathbf{w}_1} \supseteq \mathbf{D}^{\mathbf{w}_2}$	1 5 $\rightarrow$ E
11	$\mathbf{d}_1 \in \mathbf{D}^1$	9 10 Set Theory
12	$\mathbf{v_I}[\mathbf{d}_1/\mathbf{u}]\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	8 $\bigwedge$ E
13	$\mathbf{Rw}_1\mathbf{w}_2 \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	5 12 $\bigwedge$ I
14	$(\Sigma\mathbf{w}_i)(\mathbf{Rw}_1\mathbf{w}_i \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_i) = \mathbf{T})$	13 Σ I
15	$\mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\Diamond\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	14 **SR**-$\Diamond$
16	$\mathbf{d}_1 \in \mathbf{D}^1 \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}]\Diamond\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	11 15 $\bigwedge$ I
17	$(\Sigma\mathbf{d}_i)(\mathbf{d}_i \in \mathbf{D}^1 \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}]\Diamond\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T})$	16 Σ I
18	$\mathbf{v_I}((\exists\mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_1) = \mathbf{T})$	17 **SR** -$\exists$
19	$\mathbf{v_I}((\exists\mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_1) = \mathbf{T})$	7 8-18 Σ E
20	$\mathbf{v_I}((\exists\mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_1) = \mathbf{T})$	3 4-19 Σ E

This reasoning is illustrated by the following diagram, where now we represent there being at least one object $\mathbf{d}^n$ at world $\mathbf{w}^n$ by writing '$\Sigma\mathbf{d}^n$.'

$$
\begin{array}{ccc}
\mathbf{w}_1 & \overrightarrow{*} & \mathbf{w}_2 \\
\mathbf{D}^{\mathbf{w}_1} & \supseteq & \mathbf{D}^{\mathbf{w}_2} \\
\Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & & \\
\hline
\mathbf{T} & & \\
 & & (\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) \\
\hline
 & & \mathbf{T} \\
 & & \alpha(\mathbf{u})[\Sigma\mathbf{d}^2/\mathbf{u}] \\
\hline
 & & \mathbf{T} \\
 & & \alpha(\mathbf{u})[\Sigma\mathbf{d}^1/\mathbf{u}] \\
\hline
 & & \mathbf{T} \\
\Diamond\alpha(\mathbf{u})[\Sigma\mathbf{d}^1/\mathbf{u}] & & \\
\hline
\mathbf{T} & & \\
(\exists\mathbf{u})\Diamond\alpha(\mathbf{x}/\mathbf{u}) & & \\
\hline
\mathbf{T} & &
\end{array}
$$

1.5 The $\exists - \Diamond$ Form of the Converse Barcan Consequences

The final consequence to be considered is the version of the Converse Barcan Consequences which is symbolized using the '$\exists$' quantifier and '$\Diamond$' operator: $\{(\exists\mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RCI-K} \Diamond(\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$.

To prove: $\{\Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})\} \vDash_{Q1RBI-K} (\exists \mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u})$

1	$\mathbf{Rw}_1\mathbf{w}_2 \rightarrow \mathbf{D}^{\mathbf{w}_1} \subseteq \mathbf{D}^{\mathbf{w}_2}$	Includes
2	$\mathbf{v_I}((\exists \mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	Assumption
3	$(\Sigma \mathbf{d}_i)(\mathbf{d}_i \in \mathbf{D}^1 \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}]\Diamond\alpha(\mathbf{u}), \mathbf{w}_1) = \mathbf{T})$	2 **SR**-$\exists$
4	$\mathbf{d}_1 \in \mathbf{D}^1 \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\Diamond\alpha(\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	Assumption
5	$\mathbf{v_I}[\mathbf{d}_1/\mathbf{u}]\Diamond\alpha(\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	4 $\bigwedge$ E
6	$(\Sigma \mathbf{w}_i)(\mathbf{Rw}_1\mathbf{w}_i \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_i) = \mathbf{T})$	5 **SR**-$\Diamond$
7	$\mathbf{Rw}_1\mathbf{w}_2 \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	Assumption
8	$\mathbf{Rw}_1\mathbf{w}_2$	7 $\bigwedge$ E
9	$\mathbf{d}_1 \in \mathbf{D}^1$	4 $\bigwedge$ E
10	$\mathbf{D}^{\mathbf{w}_1} \subseteq \mathbf{D}^{\mathbf{w}_2}$	1 8 $\rightarrow$ E
11	$\mathbf{d}_1 \in \mathbf{D}^2$	9 Set Theory
12	$\mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	7 $\bigwedge$ E
13	$\mathbf{d}_1 \in \mathbf{D}^2 \bigwedge \mathbf{v_I}[\mathbf{d}_1/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	11 12 $\bigwedge$ I
14	$(\Sigma \mathbf{d}_i)(\mathbf{d}_i \in \mathbf{D}^2 \bigwedge \mathbf{v_I}[\mathbf{d}_i/\mathbf{u}](\alpha(\mathbf{u}), \mathbf{w}_2) = \mathbf{T})$	13 Σ I
15	$\mathbf{v_I}((\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	14 **SR**-$\exists$
16	$\mathbf{Rw}_1\mathbf{w}_2 \bigwedge \mathbf{v_I}((\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_2) = \mathbf{T}$	8 15 $\bigwedge$ I
17	$(\Sigma \mathbf{w}_i)(\mathbf{Rw}_1\mathbf{w}_i \bigwedge \mathbf{v_I}((\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_i) = \mathbf{T})$	16 Σ I
18	$\mathbf{v_I}(\Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	18 **SR** -$\Diamond$
19	$\mathbf{v_I}(\Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	6 7-18 $\exists$ E
20	$\mathbf{v_I}(\Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}), \mathbf{w}_1) = \mathbf{T}$	2 3-19 $\exists$ E

Once again, the reasoning can be illustrated with a diagram.

$$
\begin{array}{ccc}
\mathbf{w}_1 & \overset{\longrightarrow}{*} & \mathbf{w}_2 \\
\mathbf{D}^{\mathbf{w}_1} & \supseteq & \mathbf{D}^{\mathbf{w}_2} \\
(\exists \mathbf{x})\Diamond\alpha(\mathbf{x}/\mathbf{u}) & & \\ \hline
\mathbf{T} & & \\
\Diamond\alpha(\mathbf{u})[\Sigma\mathbf{d}^1/\mathbf{u}] & & \\ \hline
\mathbf{T} & & \\
 & & \alpha(\mathbf{u})[\Sigma\mathbf{d}^1/\mathbf{u}] \\ \hline
 & & \mathbf{T} \\
 & & \alpha(\mathbf{u})[\Sigma\mathbf{d}^2/\mathbf{u}] \\ \hline
 & & \mathbf{T} \\
 & & (\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) \\ \hline
 & & \mathbf{T} \\
\Diamond(\exists\mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & & \\ \hline
\mathbf{T} & &
\end{array}
$$

2 The Derivational Systems *Q1RBD-x* and *Q1RCD-x*

Giving appropriate derivational systems corresponding to the semantical systems just presented will require some new rules of inference. We will begin our treatment with the derivational systems **Q1RCD-x**, and we will then use them as models in constructing rules for *Q1RBD-x*.

We shall allow all inferences sanctioned by the *Q1RD-x* rules, with two exceptions, to be noted below. Use of the rules in the stronger derivational system will retain the reference to *Q1R*. The derivational systems *Q1RCD-x* and *Q1RBD-x* will have new rules which allow for a change in the index of the parameter, depending on the context in which it occurs. Other rules will be added to relax the restrictions on reiterating across barriers and on ending them.

To make these rules work properly, we will have to be more sensitive in our treatment of parameters, keeping track of how they were introduced: either by Universal or Existential Elimination. A parameter resulting from the use of Universal Elimination will be underlined, while a parameter that results from the use of Existential Introduction will get a line above it.[2]

2.1 The Derivational System *Q1RCD-x*

The goal in laying down rules for the derivational systems is to reflect the semantical entailments of the semantical systems. The central feature of the systems *Q1RCI-x* is the Included In condition, according to which every object in the domain of a given world is a member of the domain of each accessible world. The rules of inference for *Q1RCD-x* will reflect this semantical relation.

2.1.1 Rules for the $\Box - \forall$ Form of the Converse Barcan Consequences

To motivate the rules, we will present derivations that fail in *Q1RD-K* without them.

[2]This technique was used is a similar way for the universal quantifier in Paul Teller's *A Modern Formal Language Primer*.

Attempt to prove: $\{\Box(\forall x)Fx\} \vdash_{Q1RD-K} (\forall x)\Box Fx$

1		$\Box(\forall x)Fx$	Assumption
2	u^0	$\Box(\forall x)Fx$	1 Reiteration
3	$\Box^1$	$(\forall x)Fx$	2 SR-$\Box$
4	u^1	$(\forall x)Fx$	3 Reiteration
5		$F\underline{u^1}$	4 $\forall$ E (*Q1R*)
6		$F\underline{u^1}$	5 Misapplication of BR (*Q1R)*
7		$\Box F\underline{u^1}$	$\Box$ I
8		$(\forall x)\Box Fx$	4 Misapplication of $\forall$ I (*Q1R*)

The two problems lie in steps 6 and 8. To circumvent them, we will state two new rules. The first is that the index on a parameter '$\underline{\mathbf{u}^n}$' in a sentence that is in the scope of a barrier with an n flag (where $n > 0$) may be reduced to $n - 1$. This reflects the fact that by Included In, whatever holds of an arbitrary object in the domain of an accessible world holds for an arbitrary objects in the home domain. The second rule will relax Barrier Removal to allow such a formula to be brought out from behind the barrier. The idea here is that there should be no barrier to information which applies to an arbitrary object in both the domain of the accessible world and in that of the home world.

Index Decrement for '$\forall$' (*Q1RC)*

$\Box^n$	$\alpha(\underline{\mathbf{u}^n})$	Given
	$\vdots$	
	$\alpha(\underline{\mathbf{u}^{n-1}})$	ID $\forall$ (*Q1RC)*

Barrier Removal for '$\forall$' (*Q1RC*)

$\mathbf{u}^n$	$\vdots$	
	$\alpha(\underline{\mathbf{u}^{n-1}})$	
	$\alpha(\underline{\mathbf{u}^{n-1}})$	BR $\forall$ (*Q1RC)*

Provided that: Except for $\underline{\mathbf{u}^{n-1}}$, **u** does not occur in α,
α does not lie in the scope of any undischarged assumption.

The underlining is done to distinguish this rule from rules involving the existential quantifier. When Existential Instantiation is used, we want to be careful not to be able to generalize universally on the results. So the underlining serves to indicate when Universal Generalization is permissible.

With these rules in hand, we may now give a derivation that utilizes them..

To prove: $\{\Box(\forall x)Fx\} \vdash_{Q1RCD-K} (\forall x)\Box Fx$

1	$\Box(\forall x)Fx$	Assumption
2	$u^0 \mid\ \Box(\forall x)Fx$	1 Reiteration
3	$u^0 \mid\ \Box^1 \mid\ (\forall x)Fx$	2 SR-$\Box$
4	$u^0 \mid\ \Box^1 \mid\ u^1 \mid\ (\forall x)Fx$	3 Reiteration
5	$u^0 \mid\ \Box^1 \mid\ u^1 \mid\ F\underline{u^1}$	4 $\forall$ E (*Q1R*)
6	$u^0 \mid\ \Box^1 \mid\ u^1 \mid\ F\underline{u^0}$	5 ID $\forall$
7	$u^0 \mid\ \Box^1 \mid\ F\underline{u^0}$	6 BR $\forall$ *(Q1RC)*
8	$u^0 \mid\ \Box F\underline{u^0}$	2 3-7 $\Box$ I
9	$(\forall x)\Box Fx$	8 $\forall$ I (*Q1R*)

Note how the derivation parallels the diagrammatic illustration of the corresponding semantical entailment.

$\mathbf{w}_1$	$\xrightarrow{*}$	$\mathbf{w}_2$
$\mathbf{D}^{\mathbf{w}_1}$	$\subseteq$	$\mathbf{D}^{\mathbf{w}_2}$
$\Box(\forall\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$		
$\mathbf{T}$		
		$(\forall\mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$
		$\mathbf{T}$
		$\alpha(\mathbf{u})\ [\Pi\mathbf{d}^2/\mathbf{u}]$
		$\mathbf{T}$
		$\alpha(\mathbf{u})\ [\Pi\mathbf{d}^1/\mathbf{u}]$
		$\mathbf{T}$
$\Box\alpha(\mathbf{u})\ [\Pi\mathbf{d}^1/\mathbf{u}]$		
$\mathbf{T}$		
$(\forall\mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u})$		
$\mathbf{T}$		

2.1.2 Rules for the $\exists - \Diamond$ Form of the Converse Barcan Consequences

Now we turn to the '$\exists - \Diamond$ version of the Converse Barcan Consequences. Here is an attempt to prove the result in *Q1RD-K*

Attempt to prove: $\{(\exists x)\Diamond Fx\} \vdash_{Q1RD-K} \Diamond(\exists x)Fx$

1	$(\exists x)\Diamond Fx$	Assumption
2	$\quad \Diamond F\overline{u^0}$	Assumption
3	$\quad \Box^1 \quad F\overline{u^0}$	2 SR-$\Diamond$
4	$\quad\quad u^1 \quad F\overline{u^0}$	3 Misapplication of Reiteration *(Q1R)*
5	$\quad\quad (\exists x)Fx$	4 Misapplication of $\exists$ I (*Q1R*)
6	$\quad \Diamond(\exists x)Fx$	2 3-5 $\Diamond$ E
7	$\Diamond(\exists x)Fx$	1 2-5 $\exists$ E

The problems here is that 'u^0' may not be reiterated across a barrier and the parameter in step 4 has the wrong index for Existential Generalization. This can be solved by a version of Index Increment and a rule for Reiteration across a barrier.

Index Increment for '$\exists$' (*Q1RC)*

$$
\begin{array}{l|l}
\Box^n & \vdots \\
 & \alpha(\overline{\mathbf{u}^{n-1}}/\mathbf{x}) \\
 & \vdots \\
 & \alpha(\overline{\mathbf{u}^{n}}/\mathbf{x}) \qquad \text{II } \exists
\end{array}
$$

Barrier Crossing for $\exists$ (*Q1RC*)

$$
\begin{array}{l}
\alpha(\overline{\mathbf{u}^{n-1}}) \\
\quad \vdots \\
\mathbf{u}^n \quad \vdots \\
\quad\quad \alpha(\overline{\mathbf{u}^{n-1}}) \qquad \text{BC } \exists \ (Q1RC)
\end{array}
$$

We now prove the result for the '$\exists - \Diamond$ version of the Converse Barcan Consequences.

To prove: $\{(\exists x)\Diamond Fx\} \vdash_{Q1RCD-K} \Diamond(\exists x)Fx$

1	$(\exists x)\Diamond Fx$	Assumption
2	$\quad \Diamond F\overline{u^0}$	Assumption
3	$\quad \Box^1 \quad F\overline{u^0}$	2 Strict Assumption
4	$\quad\quad u^1 \quad F\overline{u^0}$	3 BC (*Q1RC)*
5	$\quad\quad\quad F\overline{u^1}$	4 II ∃ (*Q1RC)*
6	$\quad\quad\quad (\exists x)Fx$	5 ∃ I (*Q1R*)
7	$\quad\quad (\exists x)Fx$	6 BR (*Q1R*)
8	$\quad \Diamond(\exists x)Fx$	2 3-7 ◊ E
9	$\Diamond(\exists x)Fx$	1 2-8 ∃ E

Once again, the reasoning closely parallels what is exhibited in the corresponding semantical diagram.

$$
\begin{array}{ccc}
\mathbf{w}_1 & \overset{\longrightarrow}{*} & \mathbf{w}_2 \\
\mathbf{D}^{\mathbf{w}_1} & \supseteq & \mathbf{D}^{\mathbf{w}_2} \\
(\exists \mathbf{u})\Diamond\alpha(\mathbf{x}/\mathbf{u}) & & \\
\hline
\mathbf{T} & & \\
\Diamond\alpha(\mathbf{u})[\Sigma\mathbf{d}^1/\mathbf{u}] & & \\
\hline
\mathbf{T} & & \\
 & & \alpha(\mathbf{u})[\Sigma\mathbf{d}^1/\mathbf{u}] \\
\hline
 & & \mathbf{T} \\
 & & \alpha(\mathbf{u})[\Sigma\mathbf{d}^2/\mathbf{u}] \\
\hline
 & & \mathbf{T} \\
 & & (\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) \\
\hline
 & & \mathbf{T} \\
\Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & & \\
\hline
\mathbf{T} & &
\end{array}
$$

2.1.3 Summary of Rules for *Q1RCD-x*

Index Decrement for '∀'

Index Increment for '∃'

Barrier Removal for '∀'

Barrier Crossing for '∃'

2.2 The Derivational Systems *Q1RBD-x*

The derivational systems for the Converse Barcan Consequences are constructed from fairly simple additional rules. We can obtain rules for the system with the Barcan Consequences by a systematic change in the rules which reflects the reversal of the inclusion relation in the semantics. That is, the Index Decrement rule for '∀' becomes the Index Decrement rule for '∃,' and so on. For the Barrier Crossing/Removal rules, what is a Crossing rule for one quantifier in *Q1RCD-x* is the Removal rule for *Q1RBD-x*, and *vice-versa*. We will in this section state the rules and provide the derivations and their semantical counterparts without commentary.

2.2.1 Rules for the $\forall - \Box$ Form of the Barcan Consequences

Index Increment for '∀' (*Q1RB*)

$\Box^n$	$\alpha(\underline{\mathbf{u}^n})$	Given
	$\vdots$	
	$\alpha(\underline{\mathbf{u}^{n+1}})$	II ∀ (*Q1RB*)

Barrier Crossing for ∀ (*Q1RB*)

$\alpha(\underline{\mathbf{u}^{n-1}})$			
	$\vdots$		
	$\mathbf{u}^n$		
		$\vdots$	
		$\alpha(\underline{\mathbf{u}^{n-1}})$	BC ∀ (*Q1RB*)

To prove: $\{\Box(\forall x)Fx\} \vdash_{Q1RBD-K} (\forall x)\Box Fx$

1	$(\forall x)\Box Fx$					Assumption
2	u^0	$(\forall x)\Box Fx$				1 Reiteration
3		$\Box F\underline{u^0}$				2 ∀ E (*Q1R*)
4		$\Box^1$	$F\underline{u^0}$			3 SR-□
5			u^1	$F\underline{u^0}$		4 BC ∀ (*Q1RB*)
6				$F\underline{u^1}$		5 II ∀ (*Q1RB*)
7			$(\forall x)Fx$			6 ∀ I (*Q1R*)
8		$\Box(\forall x)Fx$				1 2-6 □ I
9	$\Box(\forall x)Fx$					7 BR

$$
\begin{array}{ccc}
\mathbf{w}_1 & \xrightarrow{*} & \mathbf{w}_2 \\
\mathbf{D}^{\mathbf{w}_1} & \supseteq & \mathbf{D}^{\mathbf{w}_2} \\
(\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u}) & & \\
\hline
\mathbf{T} & & \\
\Box\alpha(\mathbf{u})\ [\Pi\mathbf{d}^1/\mathbf{u}] & & \\
\hline
\mathbf{T} & & \\
 & & \alpha(\mathbf{u})[\Pi\mathbf{d}^1/\mathbf{u}] \\
\hline
 & & \mathbf{T} \\
 & & \alpha(\mathbf{u})[\Pi\mathbf{d}^2/\mathbf{u}] \\
\hline
 & & \mathbf{T} \\
 & & (\forall \mathbf{x})\Box\alpha(\mathbf{x}/\mathbf{u}) \\
\hline
 & & \mathbf{T} \\
\Box(\forall \mathbf{x})\alpha(\mathbf{x}/\mathbf{u}) & & \\
\hline
\mathbf{T} & &
\end{array}
$$

2.2.2 Rules for the $\Diamond - \exists$ Form of the Barcan Consequences

Index Decrement for '$\exists$' (*Q1RB)*

$$
\begin{array}{l|ll}
\Box^n & \alpha(\overline{\mathbf{u}^n}) & \text{Given} \\
 & \vdots & \\
 & \alpha(\overline{\mathbf{u}^{n-1}}) & \text{ID } \exists\ (\mathit{Q1RB})
\end{array}
$$

Barrier Removal for '$\exists$' (*Q1RB*)

$$
\begin{array}{|l}
\begin{array}{l|ll}
\mathbf{u}^n & \vdots & \\
 & \alpha(\overline{\mathbf{u}^{n-1}}) & \text{BR } \exists\ (\mathit{Q1RB})
\end{array} \\
\alpha(\overline{\mathbf{u}^{n-1}})
\end{array}
$$

Provided that: Except for $\overline{\mathbf{u}^{n-1}}$, $\mathbf{u}$ does not occur in α,
α does not lie in the scope of any undischarged assumption.

To prove: $\{\Diamond(\exists x)Fx\} \vdash_{Q1RD-K} (\exists x)\Diamond Fx$

1	$\Diamond(\exists x)Fx$	Assumption
2	u^0 | $\Diamond(\exists x)Fx$	1 Reiteration
3	u^0 | $\Box^1$ | $(\exists x)Fx$	Strict Assumption
4	u^0 | $\Box^1$ | u^1 | $F\overline{u^1}$	Assumption
5	u^0 | $\Box^1$ | u^1 | $F\overline{u^0}$	4 ID $\exists$ (*Q1RB)*
6	u^0 | $\Box^1$ | $F\overline{u^0}$	4 BR $\exists$ (*Q1RB*)
7	u^0 | $\Diamond F\overline{u^0}$	1 3-6 $\Diamond$ E
8	u^0 | $(\exists x)\Diamond Fx$	7 $\exists$ I (*Q1R*)
9	$(\exists x)\Diamond Fx$	8 BR (*Q1R*)

$\mathbf{w}_1$	$\overrightarrow{*}$	$\mathbf{w}_2$
$\mathbf{D}^{\mathbf{w}_1}$	$\supseteq$	$\mathbf{D}^{\mathbf{w}_2}$
$\Diamond(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$		
$\mathbf{T}$		
		$(\exists \mathbf{x})\alpha(\mathbf{x}/\mathbf{u})$
		$\mathbf{T}$
		$\alpha(\mathbf{u})[\Sigma\mathbf{d}^2/\mathbf{u}]$
		$\mathbf{T}$
		$\alpha(\mathbf{u})[\Sigma\mathbf{d}^1/\mathbf{u}]$
		$\mathbf{T}$
$\Diamond\alpha(\mathbf{u})[\Sigma\mathbf{d}^1/\mathbf{u}]$		
$\mathbf{T}$		
$(\exists \mathbf{u})\Diamond\alpha(\mathbf{x}/\mathbf{u})$		
$\mathbf{T}$		

2.2.3 Modification of Basic Modal Rules of Inference

The rules as they stand are too strong, in that they allow the derivation of two non-consequences of the semantical system. Specifically, $\{(\forall x)\Diamond Fx\} \nvdash_{Q1RD-S5} \Diamond(\forall x)Fx$ and $\{\Box(\exists x)Fx\} \nvdash_{Q1RD} (\exists x)\Box Fx$. Here we give a counter-example in *Q1RI-K* showing the first result, which is an adaptation of a case due to Kripke.[3] We will suppose that the frame contains three worlds, w_1, w_2, and w_3, and that $\mathrm{Rw}_1\mathrm{w}_2$ and $\mathrm{Rw}_1\mathrm{w}_3$. We let the domain at all three worlds be the $\{1, 2\}$. The extension of 'F' at w_2 is $\{\langle 1\rangle\}$ and at w_3 $\{\langle 2\rangle\}$.

On this interpretation, $\mathrm{v_I}[1/u](Fu, w_2) = \mathbf{T}$ and $\mathrm{v_I}[2/u](Fu, w_3) = \mathbf{T}$. Therefore, '$(\exists x)Fx$' is true at both those worlds, in which case '$\Box(\exists x)Fx$' is true at w_1. On the other hand, we have $\mathrm{v_I}[1/u](Fu, w_3) = \mathbf{F}$ and $\mathrm{v_I}[2/u](Fu, w_2) = \mathbf{F}$. Therefore, $\mathrm{v_I}[1/u](\Box Fu, w_1) = \mathbf{F}$ and $\mathrm{v_I}[2/u](\Box Fu, w_1) = \mathbf{F}$. Since 1 and 2 are the only two objects that exist in the domain of w_1, it follows that $\mathrm{v_I}((\exists x)\Box Fx, w_1) = \mathbf{F}$.

[3]The underlying modal system *KI* is used for clarity of exposition and to motivate the modification of a derivational rule to block the result. It can easily be adapted to *S5I*.

		$\xrightarrow{*}$	
$\mathbf{w}_1$	$\xrightarrow{*}$	$\mathbf{w}_2$	$\mathbf{w}_3$
		F: $\{\langle 1\rangle\}$	F: $\{\langle 2\rangle\}$
		$Fu[1/u]$	$Fu[2/u]$
		T	**T**
		$Fu[2/u]$	$Fu[1/u]$
		F	**F**
		$(\exists x)Fx$	$(\exists x)Fx$
		T	**T**
$\Box(\exists x)Fx$			
T			
$\Box Fu[1/u]$			
F			
$\Box Fu[2/u]$			
F			
$(\exists x)\Box Fx$			
F			

We now illustrate in a diagram the application of the same counter-example to the other case.

		$\xrightarrow{*}$	
$\mathbf{w}_1$	$\xrightarrow{*}$	$\mathbf{w}_2$	$\mathbf{w}_3$
		F: $\{\langle 1\rangle\}$	F: $\{\langle 2\rangle\}$
		$Fu[1/u]$	$Fu[2/u]$
		T	**T**
		$Fu[2/u]$	$Fu[1/u]$
		F	**F**
		$(\forall x)Fx$	$(\forall x)Fx$
		F	**F**
$\Diamond Fu[1/u]$			
T			
$\Diamond Fu[2/u]$			
T			
$(\forall x)\Diamond Fx$			
T			
$\Diamond(\forall x)Fx$			
F			

Now consider the following derivations, which are correct given the rules that have been laid down thus far.

To prove: $\{\Box(\forall x)Fx\} \vdash_{Q1RBD-K} (\forall x)\Box Fx$

1	$(\forall x)\Diamond Fx$	Assumption
2	u^0 $(\forall x)\Diamond Fx$	1 Reiteration
3	$\Diamond F\underline{u^0}$	2 $\forall$ E (*Q1R*)
4	$\Box^1$ $F\underline{u^0}$	Strict Assumption
5	u^1 $F\underline{u^0}$	4 BC $\forall$ (*Q1RB)*
6	$F\underline{u^1}$	5 II $\forall$ (*Q1RB)*
7	$(\forall x)Fx$	6 $\forall$ I (*Q1R*)
8	$\Diamond(\forall x)Fx$	1 2-6 W $\Diamond$ I
9	$\Diamond(\forall x)Fx$	7 BR

Attempt to prove: $\{\Box(\exists x)Fx\} \vdash_{Q1RD-K} (\exists x)\Box Fx$

1	$\Box(\exists x)Fx$	Assumption
2	u^0 $\Box(\exists x)Fx$	1 Reiteration
3	$\Box^1$ $(\exists x)Fx$	2 SR-$\Box$
4	u^1 $F\overline{u^1}$	Assumption
5	$F\overline{u^0}$	4 ID $\exists$ (*Q1RB)*
6	$F\overline{u^0}$	4 BR $\exists$ (*Q1RB*)
7	$\Box F\overline{u^0}$	3-6 $\Box$ I
8	$(\exists x)\Box Fx$	7 $\exists$ I (*Q1R*)
9	$(\exists x)\Box Fx$	8 BR (*Q1R*)

The problem with the first derivation lies at step 4, where a Strict Assumption is made. In this context, the parameter u^0 loses its arbitrariness, since (as the counter-example shows), it might stand for different existing things at different worlds. Thus, we will amend the rule for Strict Assumption in the underlying modal system. Since the problem does not arise for parameters indexed to the current restricted scope line, we will only ban the Strict Assumption of a sentence with a parameter whose index is one less than the current restricted scope line.

Strict Assumption (Q1RB)

$\Diamond\alpha$	Already Derived
$\Box^n$ α	SR-$\Diamond$ (*Q1RB)*
$\vdots$	

Provided that α is strictly reiterated across exactly one restricted scope line;
$\mathbf{u}^{n-1}$ does not occur in α.

In the second case, the problem lies at step 7. Once again, the problem is lack of arbitrariness. It may not be that u^0 represents the same object in the domains of each of the accessible worlds. Accordingly, we will modify the rule of $\Box$ Introduction.

$\Box$ Introduction (Q1RB)

$$\begin{array}{l|l} \Box^n & \vdots \\ & \\ & \alpha \end{array}$$

$$\Box\alpha \qquad \Box\,\mathrm{I}$$

Provided that α is not to the right of any scope line;
$\mathbf{u}^{n-1}$ does not occur in α.

Finally, it should be noted that these problems do not arise in the *Q1RCD-x* systems. These systems lack the rules for changing the index or breaching the barrier which are used in the two derivations above. This makes sense, since the two non-entailments are closely related to the two Barcan Consequences, $\{(\forall x)\Box Fx\} \vdash_{Q1RBD-K} \Box(\forall x)Fx$ and $\{\Diamond(\exists x)Fx\} \vdash_{Q1RBD} (\exists x)\Diamond Fx$.

2.2.4 Summary of Rules for *Q1RBD-x*

Index Increment for ‘$\forall$’

Index Decrement for ‘$\exists$’

Barrier Crossing for ‘$\forall$’

Barrier Removal for ‘$\exists$’

Strict Assumption (*Q1RD-x*)

$\forall$ Introduction (*Q1RD-x*)